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ON BISUBDIFFERENTIALS OF BICONVEX OPERATORS

Abstract

In the work bisubdifferentials and biadjoint operator for biconvex operators are defined, and some of their properties are studied.

In the work bisubdifferentials and biadjoint operator for biconvex operators and defined and a number of their properties are studied.

Let E be some ordered vector space, i.e. the space with preferred salient (convex) cone E^+, a cone of positive elements. Let's join to the space E the greatest element $+\infty$, at that the order, induced E from $\bar{E} = E \cup \{+\infty\}$ coincides with initial order in E (see [1]).

The ordered real vector space E, which is simultaneaously a lattice is called a vector lattice. The vector lattice, in which any ordinal bounded set has exact boundaries is called Kontarovich space, or in short K space.

Let X and Y be real vector spaces. The mapping $f : X \times Y \to \bar{E}$ is called biconvex (see [2]), if the mappings $f(\cdot, y) : X \to \bar{E}$, $f(x, \cdot) : Y \to \bar{E}$ are convex at $x \in X$, $y \in Y$. Denote by $\tilde{B}(X \times Y, E)$ the vector space of bilinear operators from $X \times Y$ in E. Let $f : X \times Y \to \bar{E}$ be a mapping and $domf = \{(x, y) \in X \times Y : f(x, y) < +\infty\}$. It is easy to check, that if f is a biconvex operator, then $domf$ is a biconvex set. The following set

$$\partial_2^a f(\bar{x}, \bar{y}) = \{x^* \in \tilde{B}(X \times Y, R) : f(x, y) - f(\bar{x}, \bar{y}) \geq$$

$$\geq x^*(x, y) - x^*(\bar{x}, \bar{y}), (x, y) \in X \times Y\}$$

we'll call the bisubdifferential of the operator f at the point $(\bar{x}, \bar{y}) \in domf$ and if $P : X \times Y \to \bar{E}$ is a bisublinear operator, then suppose, that

$$\partial_2^a P = \{x^* \in \tilde{B}(X \times Y, R) : P(x, y) \geq x^*(x, y), (x, y) \in X \times Y\}.$$

Lemma 1. *If $P : X \times Y \to \bar{E}$ is a bisublinear operator, $P(0, 0) = 0$ and $(\bar{x}, \bar{y}) \in domP$, then*

$$\partial_2^a P(\bar{x}, \bar{y}) = \{x^* \in \tilde{B}(X \times Y, E) : x^* \in \partial_2^a P, P(\bar{x}, \bar{y}) = x^*(\bar{x}, \bar{y})\}.$$

Proof. If $x^* \in \partial_2^a P(\bar{x}, \bar{y})$, then

$$P(x, y) - P(\bar{x}, \bar{y}) \geq x^*(x, y) - x^*(\bar{x}, \bar{y}), \ (x, y) \in X \times Y.$$

Assuming $(x, y) = (0, 0)$ and $(x, y) = (2\bar{x}, \bar{y})$ respectively, we'll obtain, that $P(\bar{x}, \bar{y}) \leq x^*(\bar{x}, \bar{y})$ and $2P(\bar{x}, \bar{y}) - P(\bar{x}, \bar{y}) \geq 2x^*(\bar{x}, \bar{y}) - x^*(\bar{x}, \bar{y})$. Hence $P(\bar{x}, \bar{y}) = x^*(\bar{x}, \bar{y})$. Then we have $P(x, y) \geq x^*(x, y)$ at $(x, y) \in X \times Y$, i.e. $x^* \in \partial_2^a P$.

Vice verse, if $x^* \in \partial_2^a P$ and $P(\bar{x}, \bar{y}) = x^*(\bar{x}, \bar{y})$, then

$$P(x, y) - P(\bar{x}, \bar{y}) \geq x^*(x, y) - x^*(\bar{x}, \bar{y}), \ (x, y) \in X \times Y, \text{ i.e. } x^* \in \partial_2^a P(\bar{x}, \bar{y}).$$

The lemma is proved.

Lemma 2. *If* $x^* \in \partial_2^a f(\bar{x}, \bar{y})$, *then*

$$f(\bar{x} + x, \bar{y} + y) - 2f(\bar{x}, \bar{y}) + f(\bar{x} - x, \bar{y} - y) \geq 2x^*(x, y), (x, y) \in X \times Y.$$

Proof. From the definition $\partial_2^a f(\bar{x}, \bar{y})$ it implies, that

$$f(\bar{x} + x, \bar{y} + y) - f(\bar{x}, \bar{y}) \geq x^*(\bar{x} + x, \bar{y} + y) - x^*(\bar{x}, \bar{y}),$$

$$f(\bar{x} - x, \bar{y} - y) - f(\bar{x}, \bar{y}) \geq x^*(\bar{x} - x, \bar{y} - y) - x^*(\bar{x}, \bar{y}),$$

at $(x, y) \in X \times Y$. Adding these correlations we have

$$f(\bar{x} + x, \bar{y} + y) - 2f(\bar{x}, \bar{y}) + f(\bar{x} - x, \bar{y} - y) \geq$$

$$\geq x^*(\bar{x} + x, \bar{y} + y) - 2x^*(\bar{x}, \bar{y}) + x^*(\bar{x} - x, \bar{y} - y) = 2x^*(x, y).$$

The lemma is proved.

For simplicity we'll assume that E is K space.

Let $x^* \in \tilde{B}(X \times Y, E)$, $f : X \times Y \to \bar{E}$. Suppose

$$f^*(x^*) = \sup_{\substack{x \in X \\ y \in Y}} \{x^*(x, y) - f(x, y)\}.$$

Lemma 3. *If* f *is a biconvex operator, then* $x^* \in \partial_2^a f(\bar{x}, \bar{y})$ *if and only if*

$$f^*(x^*) + f(\bar{x}, \bar{y}) = x^*(\bar{x}, \bar{y}).$$

Proof. If $x^* \in \partial_2^a f(\bar{x}, \bar{y})$, then

$$x^*(\bar{x}, \bar{y}) - f(\bar{x}, \bar{y}) \geq x^*(x, y) - f(x, y)$$

as $(x, y) \in X \times Y$. Therefore

$$x^*(\bar{x}, \bar{y}) - f(\bar{x}, \bar{y}) = \sup_{\substack{x \in X \\ y \in Y}} \{x^*(x, y) - f(x, y)\},$$

i.e. $f^*(x^*) + f(\bar{x}, \bar{y}) = x^*(\bar{x}, \bar{y})$. Inversely, if

$$f^*(x^*) + f(\bar{x}, \bar{y}) = x^*(\bar{x}, \bar{y}),$$

then

$$\sup_{\substack{x \in X \\ y \in Y}} \{x^*(x, y) - f(x, y)\} = x^*(\bar{x}, \bar{y}) - f(\bar{x}, \bar{y}).$$

Therefore $x^*(x, y) - f(x, y) \leq x^*(\bar{x}, \bar{y}) - f(\bar{x}, \bar{y})$ at $(x, y) \in X \times Y$, i.e. $f(x, y) - f(\bar{x}, \bar{y}) \geq x^*(x, y) - x^*(\bar{x}, \bar{y})$ at $(x, y) \in X \times Y$.

The lemma is proved.

Let $g : X \times Y \to \bar{E}$ and $domg = \{(x, y) \in X \times Y : g(x, y) < +\infty\} \neq \varnothing$. As for each pair $(x, y) \in X \times Y$, the mapping $x^* \to x^*(x, y)$ is a linear operator on $\tilde{B}(X \times Y, E)$, then from the definition of $g^*(x^*)$, it implies, that $x^* \to g^*(x^*)$ is a convex operator.

Suppose

$$g^{**}(x, y) = \sup_{x^* \in \tilde{B}(X \times Y, E)} \{x^*(x, y) - g^*(x^*)\}.$$

It is clear, that $(x,y)\to x^*(x,y)-g^*(x^*)$ is a biconvex operator. Therefore from p. 1.3.6 (1) [1] it implies, that $(x,y)\to g^{**}(x,y)$ is a biconvex operator. From the definition of $g^*(x^*)$ we'll obtain, that $g^*(x^*)\geq x^*(x,y)-g(x,y)$ at $(x,y)\in X\times Y$ and $x^*\in\tilde{B}(X\times Y,E)$. Therefore $g^{**}(x,y)\leq g(x,y)$ for any $(x,y)\in X\times Y$.

Let $f:X\times Y\to\bar{E}$ be a bipositive homogeneous operator. Suppose

$$\partial_2^a f=\{x^*\in\tilde{B}(X\times Y,E):f(x,y)\geq x^*(x,y),\ (x,y)\in X\times Y\}.$$

Lemma 4. *If $f:X\times Y\to\bar{E}$ is a bipositive homogeneous operator and $f(0,0)=0$, then*

$$f^*(x^*)=\begin{cases}0; & x^*\in\partial_2^a f\\ +\infty; & x^*\notin\partial_2^a f.\end{cases}$$

Proof. If $x^*\in\partial_2^a f$, then

$$f^*(x^*)=\sup_{(x,y)\in X\times Y}\{x^*(x,y)-f(x,y)\}\leq 0=x^*(0,0)-f(0,0)\leq f^*(x^*).$$

Hence it implies, that $f^*(x^*)=0$.

If $x^*\notin\partial_2^a f$, then there exists $(\bar{x},\bar{y})\in X\times Y$ such that $f(\bar{x},\bar{y})<x^*(\bar{x},\bar{y})$. Then

$$f^*(x^*)=\sup_{(x,y)\in X\times Y}\{x^*(x,y)-f(x,y)\}\geq\sup_{\lambda\geq 0}\{x^*(\lambda\bar{x},\bar{y})-f(\lambda\bar{x},\bar{y})\}=$$

$$=\sup_{\lambda\geq 0}\lambda\{x^*(\bar{x},\bar{y})-f(\bar{x},\bar{y})\}=+\infty.$$

The lemma is proved.

Theorem 1. *If $f(x,y)=\sup\limits_{x^*\in\partial_2^a f}x^*(x,y)$, then $f^{**}(x,y)=f(x,y)$.*

Proof. From lemma 4 it implies, that

$$f^*(x^*)=\begin{cases}0; & x^*\in\partial_2^a f\\ +\infty; & x^*\notin\partial_2^a f\end{cases}.$$

Therefore $f^{**}(x,y)=\sup\limits_{x^*\in\tilde{B}(X\times Y,E)}\{x^*(x,y)-f^*(x^*)=\sup\limits_{x^*\in\partial_2^a f}x^*(x,y)$, i.e. $f^{**}(x,y)=f(x,y)$. The theorem is proved.

Theorem 2. *Let $domf$ be a biconvex set and $\partial_2^a f(x,y)\neq\varnothing$ for any $(x,y)\in domf$. Then $f:X\times Y\to\bar{E}$ is a biconvex operator.*

Proof. Let $(x_1,y),(x_2,y)\in domf$, $\alpha\in[0,1]$, $\bar{x}=\alpha x_1+(1-\alpha)x_2$ and $x^*\in\partial_2^a f(\bar{x},y)$. Then we'll obtain, that

$$f(x_1,y)-f(\bar{x},y)\geq x^*(x_1,y)-x^*(\bar{x},y),$$

$$f(x_2,y)-f(\bar{x},y)\geq x^*(x_2,y)-x^*(\bar{x},y).$$

Multiplying the first of these inequalities by α, and the second by $1-\alpha$ and adding them we have

$$\alpha f(x_1,y)+(1-\alpha)f(x_2,y)-f(\bar{x},y)\geq\alpha x^*(x_1,y)+(1-\alpha)x^*(x_2,y)-$$

$$-x^*(\bar{x},y)=x^*(\alpha x_1+(1-\alpha)x_2,y)-x^*(\bar{x},y)=0.$$

Therefore

$$\alpha f(x_1,y)+(1-\alpha)f(x_2,y)\geq f(\bar{x},y)$$

at $(x_1,y),(x_2,y)\in domf$ and $\alpha\in[0,1]$. Similarly we have, that

$$\beta f(x,y_1)+(1-\beta)f(x,y_2)\geq f(x,\beta y_1+(1-\beta)y_2)$$

at $(x,y_1),(x,y_2)\in domf$ and $\beta\in[0,1]$. The theorem is proved.

The set of all bisublinear operators P from $X\times Y$ to E, which satisfy the condition $P(-x,-y)=P(x,y)$ we'll denote by H.

Let's note, that if $P_1:X\times Y\to E$ and $P_2:X\times Y\to E$, then $\partial_2^a(P_1+P_2)\supset\partial_2^aP_1+\partial_2^aP_2$.

Really, if $x_1^*\in\partial_2^aP_1,\ x_2^*\in\partial_2^aP_2$, then $P_1(x,y)\geq x_1^*(x,y),P_2(x,y)\geq x_2^*(x,y)$ at $(x,y)\in X\times Y$. Therefore $P_1(x,y)+P_2(x,y)\geq x_1^*(x,y)+x_2^*(x,y)$ at $(x,y)\in X\times Y$, i.e. $x_1^*+x_2^*\in\partial_2^a(P_1+P_2)$.

Theorem 3. *If* $P_1,P_2\in H$ *and* $\dim Y=1$, *then* $\partial_2^a(P_1+P_2)=\partial_2^aP_1+\partial_2^aP_2$.

Proof. The inclusion $\partial_2^a(P_1+P_2)\supset\partial_2^aP_1+\partial_2^aP_2$ is obvious. Let us prove the inverse inclusion. Let $x^*\in\partial_2^a(P_1+P_2)$. Define the mappings Ψ and z^*, acting from the space $(X\times X)\times(Y\times Y)$ and diagonals $\Delta(X\times Y)=\{((x,x),(y,y)):x\in X,\ y\in Y\}$ respectively, by the formulae

$$\Psi((x_1,x_2),(y_1,y_2))=P_1(x_1,y_1)+P_2(x_2,y_2),$$

$$z^*((x,x),(y,y))=x^*(x,y).$$

Then Ψ is a bisublinear operator, z^* is bilinear operator and $z^*(\vartheta,\omega)\leq\Psi(\vartheta,\omega)$ for all $(\vartheta,\omega)\in\Delta(X\times Y)$. By theorem 2.3.2 [3] there exists the bilinear operator $Z^*:(X\times X)\times(Y\times Y)\to E$ such that $Z^*\in\partial_2^a\Psi$ and the contraction Z^* on $\Delta(X\times Y)$ coincides with z^*. Suppose $z_1^*(x,y)=Z^*((x,0),(y,y)),\ z_2^*(x,y)=Z^*((0,x),(y,y))$. It is easy to check, that z_1^* and z_2^* are bilinear operators from $X\times Y$ to E and $x^*(x,y)=z_1^*(x,y)+z_2^*(x,y)$. Besides, $z_1^*(x,y)\leq\Psi((x,0),(y,y))$ $=P_1(x,y)+P_2(0,y)=P_1(x,y)$,i.e. $z_1^*\in\partial_2^aP_1$ and $z_2^*(x,y)\leq\Psi((0,x),(y,y))$ $=P_1(0,y)+P_2(x,y)=P_2(x,y)$, i.e. $z_2^*\in\partial_2^aP_2$. Therefore $\partial_2^a(P_1+P_2)\subset\partial_2^aP_1+\partial_2^aP_2$. The theorem is proved.

Theorem 4. *Let* P *and* Q *belong to* H *and* $P(x,y)+Q(x,y)\geq0$ *for all* $(x,y)\in X\times Y$ and $\dim Y=1$. *Then there exists a bilinear operator* $x^*:X\times Y\to E$ *such that*

$$-Q(x,y)\leq x^*(x,y)\leq P(x,y),\ (x,y)\in X\times Y.$$

Proof. By the condition of the theorem $0\in\partial_2^a(P+Q)$. By theorem 3 there exist $x^*\in\partial_2^aP$ and $z^*\in\partial_2^aQ$ such, that $x^*+z^*=0$. Then we have, that $-Q(x,y)\leq x^*(x,y)\leq P(x,y)$ as $(x,y)\in X\times Y$. The theorem is proved.

Let $P:X\times Y\to E$ be a bisublinear operator, X_0 and Y_0 be subspaces in X and Y, respectively. Suppose, that

$$P_{X_0\times Y_0}(x,y)=\begin{cases}P(x,y):(x,y)\in X_0\times Y_0\\ +\infty:(x,y)\notin X_0\times Y_0\end{cases},$$

$$\delta_{X_0\times Y_0}(x,y)=\begin{cases}0:(x,y)\in X_0\times Y_0\\ \\ +\infty:(x,y)\notin X_0\times Y_0\end{cases}.$$

Theorem 5. *If* $P\in H$ and $\dim Y=1$, *then* $\partial_2^a P_{X_0\times Y_0}=\partial_2^a P+\partial_2^a\delta_{X_0\times Y_0}$.

Proof. If $x_1^*\in\partial_2^a P$ and $x_2^*\in\partial_2^a\delta_{X_0\times Y_0}$, then $P(x,y)\geq x_1^*(x,y)$ and $\delta_{X_0\times Y_0}(x,y)\geq x_2^*(x,y)$ at $(x,y)\in X\times Y$. Therefore $P(x,y)+\delta_{X_0\times Y_0}(x,y)\geq x_1^*(x,y)+x_2^*(x,y)$ at $(x,y)\in X\times Y$, i.e. $\partial_2^a P+\partial_2^a\delta_{X_0\times Y_0}\subset\partial_2^a P_{X_0\times Y_0}$.

If $x^*\in\partial_2^a P_{X_0\times Y_0}$, then $P(x,y)+\delta_{X_0\times Y_0}(x,y)\geq x^*(x,y)$ at $(x,y)\in X\times Y$. Therefore $P(x,y)\geq x^*(x,y)$ at $(x,y)\in X_0\times Y_0$. Using theorems 2.3.2 [3] we have, that there exists the bilinear operator $z^*:X\times Y\to E$ such that $z^*(x,y)=x^*(x,y)$ at $(x,y)\in X_0\times Y_0$ and $P(x,y)\geq z^*(x,y)$ at $(x,y)\in X\times Y$. Then $\delta_{X_0\times Y_0}(x,y)\geq x^*(x,y)-z^*(x,y)$ at $(x,y)\in X\times Y$. Therefore $x^*\in\partial_2^a P+\partial_2^a\delta_{X_0\times Y_0}$, i.e. $\partial_2^a P_{X_0\times Y_0}\subset\partial_2^a P+\partial_2^a\delta_{X_0\times Y_0}$.

The theorem is proved.

Corollary 1. *If* $P\in H$ and $\dim Y=1$, *then* $\partial_2^a P$ *is not empty.*

Really, from theorem 3 it implies, that

$$\partial_2^a P_{(0,0)}=\partial_2^a P+\partial_2^a\delta_{(0,0)}.$$

As $\partial_2^a P_{(0,0)}$ and $\partial_2^a\delta_{(0,0)}$ are not empty hence it implies that $\partial_2^a P$ is not empty.

If X and Y are topological vector spaces, E is topological K space, then we'll denote by $B(X\times Y,E)$ the vector space of all continuous bilinear operators from $X\times Y$ to E. Let $f:X\times Y\to\bar{E}$ and $(\bar{x},\bar{y})\in domf$. Assume, that

$$\partial_2 f(\bar{x},\bar{y})=\{x^*\in B(X\times Y,E):f(x,y)-f(\bar{x},\bar{y})\geq x^*(x,y)-$$

$$-x^*(\bar{x},\bar{y}),(x,y)\in X\times Y\},$$

and if $P:X\times Y\to\bar{E}$ is a bisublinear operator, then suppose, that

$$\partial_2 P=\{x^*\in B(X\times Y,E):P(x,y)\geq x^*(x,y),(x,y)\in X\times Y\}.$$

Let $P:X\times Y\to\bar{E}, domP=\{(x,y)\in X\times Y:P(x,y)<+\infty\}$. Assume, that

$$H((x,\lambda),(y,\mu))=\begin{cases}\lambda\mu P\left(\dfrac{x}{\lambda},\dfrac{y}{\mu}\right),\lambda>0,\mu>0,\left(\dfrac{x}{\lambda},\dfrac{y}{\mu}\right)\in domP\\ \\ 0,\quad (x,\lambda)=0\quad\text{or}\quad (y,\mu)=0\\ \\ +\infty,\quad\text{other cases.}\end{cases}\tag{1}$$

Lemma 5. *If* $P:X\times Y\to\bar{E}$ *is a biconvex operator, then the operator* $H:(X\times R)\times(Y\times R)\to\bar{E}$ *defined by equality (1) is bisublinear.*

Proof. If $\left(\dfrac{x}{\lambda},\dfrac{y}{\mu}\right)\in domP,\ \lambda>0,\ \mu>0$ and $\alpha>0$, then

$$H(\alpha(x,\lambda),(y,\mu))=\alpha\lambda\mu P\left(\frac{x\alpha}{\alpha\lambda},\frac{y}{\mu}\right)=\alpha\lambda\mu P\left(\frac{x}{\lambda},\frac{y}{\mu}\right)=$$

$$=\alpha H((x,\lambda),(y,\mu)).$$

It is similarly checked, that $H((x,\lambda),\beta(y,\mu))=\beta H((x,\lambda),(y,\mu))$ at $\beta>0$, i.e. H is bipositively homogeneous.

Let $\left(\frac{x_1}{\lambda_1},\frac{y}{\mu}\right)$ and $\left(\frac{x_2}{\lambda_2},\frac{y}{\mu}\right)$ belong to $domP$ and $\lambda_1>0,\lambda_2>0$ and $\mu>0$. Then

$$H\left((x_1,\lambda_1)+(x_2,\lambda_2),(y,\mu)\right)=\mu\left(\lambda_1+\lambda_2\right)P\left(\frac{x_1+x_2}{\lambda_1+\lambda_2},\frac{y}{\mu}\right)=$$

$$=\mu\left(\lambda_1+\lambda_2\right)P\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{x_1}{\lambda_1}+\frac{\lambda_2}{\lambda_1+\lambda_2}\frac{x_2}{\lambda_2},\frac{y}{\mu}\right)\le$$

$$\le\mu\left(\lambda_1+\lambda_2\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}P\left(\frac{x_1}{\lambda_1},\frac{y}{\mu}\right)+\frac{\lambda_2}{\lambda_1+\lambda_2}P\left(\frac{x_2}{\lambda_2},\frac{y}{\mu}\right)\right)=$$

$$=\mu\lambda_1P\left(\frac{x_1}{\lambda_1},\frac{y}{\mu}\right)+\mu\lambda_2P\left(\frac{x_2}{\lambda_2},\frac{y}{\mu}\right).$$

The other cases are obvious. The lemma is proved.

If $(\bar{x},\bar{y})\in domP$ and $x^*\in\partial_2H\left((\bar{x},1),(\bar{y},1)\right)=\{x^*\in\partial_2H:H\left((\bar{x},1),(\bar{y},1)\right)=x^*\left((\bar{x},1),(\bar{y},1)\right)\}$, then $P(x,y)\ge x^*\left((x,1),(y,1)\right)$ at $(x,y)\in X\times Y$ and $P(\bar{x},\bar{y})=x^*\left((\bar{x},1),(\bar{y},1)\right)$. Therefore $P(x,y)-P(\bar{x},\bar{y})\ge x^*\left((x,1),(y,1)\right)-x^*\left((\bar{x},1),(\bar{y},1)\right)$ at $(x,y)\in X\times Y$.

For simplicity we'll assume, that the value of the operator belongs to E.

Assume, that

$$f^{(2)+}\left((\bar{x},\bar{y});(x,y)\right)=$$

$$=\overline{\lim_{\lambda\downarrow 0,\mu\downarrow 0}}\frac{1}{\lambda\mu}\left(f\left(\bar{x}+\lambda x,\bar{y}+\mu y\right)-2f\left(\bar{x},\bar{y}\right)+f\left(\bar{x}-\lambda x,\bar{y}-\mu y\right)\right).$$

Lemma 6. *If $f:X\times Y\to E$ is a biconvex operator, then*

$$(x,y)\to f^{(2)+}\left((\bar{x},\bar{y});(x,y)\right)$$

is a bisublinear operator.

Proof. It is easily checked, that at $\alpha>0$

$$f^{(2)+}\left((\bar{x},\bar{y});(\alpha x,y)\right)=\alpha f^{(2)+}\left((\bar{x},\bar{y});(x,y)\right).$$

If $x_1,x_2\in X,\ \alpha\in[0,1]$, then

$$f^{(2)+}\left((\bar{x},\bar{y});(\alpha x_1+(1-\alpha)x_2,y)\right)=\overline{\lim_{\lambda\downarrow 0,\mu\downarrow 0}}\frac{1}{\lambda\mu}(f(\alpha\left(\bar{x}+\lambda x_1\right)+$$

$$+(1-\alpha)\left(\bar{x}+\lambda x_2\right),\bar{y}+\mu y)-2f\left(\bar{x},\bar{y}\right)+f(\alpha\left(\bar{x}-\lambda x_1\right)+$$

$$+(1-\alpha)\left(\bar{x}-\lambda x_2\right),\bar{y}-\mu y))\le\alpha\overline{\lim_{\lambda\downarrow 0,\mu\downarrow 0}}\frac{1}{\lambda\mu}(f\left(\bar{x}+\lambda x_1,\bar{y}+\mu y\right)-$$

$$-2f\left(\bar{x},\bar{y}\right)+f\left(\bar{x}-\lambda x_1,\bar{y}-\mu y\right))+(1-\alpha)\overline{\lim_{\lambda\downarrow 0,\mu\downarrow 0}}\frac{1}{\lambda\mu}(f\left(\bar{x}+\lambda x_2,\bar{y}+\mu y\right)-$$

$$-2f\left(\bar{x},\bar{y}\right)+f\left(\bar{x}-\lambda x_2,\bar{y}-\mu y\right))=\alpha f^{(2)+}\left((\bar{x},\bar{y});(x_1,y)\right)+$$

$$+(1-\alpha)f^{(2)+}\left((\bar{x},\bar{y});(x_2,y)\right).$$

Therefore, we'll easily obtain the validity of Lemma 6. The lemma is proved.

Denote

$$\bar{\partial}^2f\left(\bar{x},\bar{y}\right)=\{x^*\in B\left(X\times Y,E\right):f^{(2)+}\left((\bar{x},\bar{y});(x,y)\right)\ge$$

$$\geq 2x^*(x,y)\,,\ (x,y)\in X\times Y\},$$

$$\partial^2 f(\bar{x},\bar{y})=\{x^*\in B(X\times Y,E): f(\bar{x}+x,\bar{y}+y)-2f(\bar{x},\bar{y})+$$

$$+f(\bar{x}-x,\bar{y}-y)\geq 2x^*(x,y)\,,\ (x,y)\in X\times Y\}.$$

From the definition it directly implies, that $\partial^2 f(\bar{x},\bar{y})\subset\bar{\partial}^2 f(\bar{x},\bar{y})$.

Lemma 7. *If P is busiblinear operator, then $\partial_2 P\subset\partial^2 P(0,0)$. Besides, if $P(-x,-y)=P(x,y)$, then $\partial_2 P=\partial_2 P(0,0)$.*

Proof. If $x^*\in\partial_2 P$, then $P(x,y)\geq x^*(x,y)$ and $P(-x,-y)\geq x^*(x,y)$ at $(x,y)\in X\times Y$. Therefore $P(x,y)+P(-x,-y)\geq 2x^*(x,y)$ at $(x,y)\in X\times Y$ i.e. $x^*\in\partial^2 P(0,0)$. Then we have, that $\partial_2 P\subset\partial^2 P(0,0)$.

If $P(-x,-y)=P(x,y)$ and $x^*\in\partial^2 P(0,0)$, then

$$P(x,y)+P(-x,-y)=2P(x,y)\geq 2x^*(x,y)$$

at $(x,y)\in X\times Y$. Therefore $x^*\in\partial_2 P$, i.e. $\partial^2 P(0,0)\subset\partial_2 P$. As $\partial_2 P\subset\partial^2 P(0,0)$, then we'll obtain, that $\partial_2 P=\partial^2 P(0,0)$. The lemma is proved.

Lemma 8. *If P is a bisublinear operator, then*

$$\partial^2 P(\bar{x},\bar{y})\supset\{x^*\in B(X\times Y,R): x^*\in\partial_2 P, P(\bar{x},\bar{y})=x^*(\bar{x},\bar{y})\}$$

and form $x^*\in\partial^2 P(\bar{x},\bar{y})$ it implies, that $P(\bar{x},\bar{y})=x^*(\bar{x},\bar{y})$.

Proof. If $x^*\in\partial_2 P$ and $P(\bar{x},\bar{y})=x^*(\bar{x},\bar{y})$, then

$$P(\bar{x}+y,\bar{y}+y)-2P(\bar{x},\bar{y})+P(\bar{x}-x,\bar{y}-y)\geq x^*(\bar{x}+x,\bar{y}+y)-$$

$$-2x^*(\bar{x},\bar{y})+x^*(\bar{x}-x,\bar{y}-y)=x^*(\bar{x},\bar{y})+x^*(x,y)+x^*(x,\bar{y})+x^*(\bar{x},y)-$$

$$-2x^*(\bar{x},\bar{y})+x^*(\bar{x},\bar{y})+x^*(x,y)-x^*(\bar{x},y)-x^*(x,\bar{y})=2x^*(x,y)\,,$$

i.e. $x^*\in\partial^2 P(\bar{x},\bar{y})$.

If $x^*\in\partial^2 P(\bar{x},\bar{y})$, then taking $(x,y)=(\bar{x},\bar{y})$, we'll obtain $2P(\bar{x},\bar{y})\geq 2x^*(\bar{x},\bar{y})$, i.e. $P(\bar{x},\bar{y})\geq x^*(\bar{x},\bar{y})$. Assuming $(x,y)=(-\bar{x},\bar{y})$ we have, that $-2P(\bar{x},\bar{y})\geq 2x^*(-\bar{x},\bar{y})$, i.e. $P(\bar{x},\bar{y})\leq x^*(\bar{x},\bar{y})$. Therefore $P(\bar{x},\bar{y})=x^*(\bar{x},\bar{y})$. The lemma is proved.

Corollary 2. *If P is a bisublinear operator, then*

$$\partial^2 P(\bar{x},\bar{y})=\{x^*\in B(X\times Y,R): P(\bar{x}+x,\bar{y}+y)+P(\bar{x}-x,\bar{y}-y)\geq$$

$$\geq 2x^*(x,y)+2x^*(\bar{x},\bar{y})\,,\quad(x,y)\in X\times Y\}.$$

Corollary 3. *If P is a bisublinear operator and $x^*\in\partial^2 P(\bar{x},\bar{y})$, then $P(x,\bar{y})\geq x^*(x,\bar{y})\,,P(\bar{x},\upsilon)\geq x^*(\bar{x},\upsilon)$ at $x\in X,\upsilon\in Y$.*

Proof. If $x^*\in\partial^2 P(\bar{x},\bar{y})$, by corollary 2 we have, that

$$P(\bar{x}+x,\bar{y}+y)+P(\bar{x}-x,\bar{y}-y)\geq 2x^*(x,y)+2x^*(\bar{x},\bar{y})\ \text{as}\ (x,y)\in X\times Y.$$

Assuming $y=\bar{y}$, hence we'll obtain, that $P(\bar{x}+x,2\bar{y})\geq 2x^*(x,\bar{y})+2x^*(\bar{x},\bar{y})$, i.e. $2P(\bar{x}+x,\bar{y})\geq 2x^*(\bar{x}+x,\bar{y})$ at $x\in X$. Therefore $P(z,\bar{y})\geq x^*(z,\bar{y})$. Assuming $x=\bar{x}$, similarly we have, that $P(\bar{x},\upsilon)\geq x^*(\bar{x},\upsilon)$ at $\upsilon\in Y$. The corollary is proved.

Lemma 9. *Let P be a bisublinear operator and $x^*\in B(X\times Y,E)$. Then the inequality*

$$P(\bar{x}+x,\bar{y}+y)-P(\bar{x}+x,\bar{y})-P(\bar{x},\bar{y}+y)+P(\bar{x},\bar{y})\geq$$

$$\geq x^*(x,y), (x,y) \in X \times Y \qquad (2)$$

is fulfilled if and only if, $x^* \in \partial_2 P, P(z,\bar{y}) = x^*(z,\bar{y})$ *and* $P(\bar{x},\upsilon) = x^*(\bar{x},\upsilon)$.

Proof. If (2) is fulfilled, then assuming $(x,y) = (\bar{x},\bar{y})$, we'll obtain

$$P(2\bar{x},2\bar{y}) - P(2\bar{x},\bar{y}) - P(\bar{x},2\bar{y}) + P(\bar{x},\bar{y}) \geq x^*(\bar{x},\bar{y}),$$

i.e. $P(\bar{x},\bar{y}) \geq x^*(\bar{x},\bar{y})$. Assuming $(x,y) = (-\bar{x},\bar{y})$, we also have

$$P(\bar{x}-\bar{x},2\bar{y}) - P(\bar{x}-\bar{x},\bar{y}) - P(\bar{x},2\bar{y}) + P(\bar{x},\bar{y}) \geq x^*(-\bar{x},\bar{y}),$$

i.e. $-P(\bar{x},\bar{y}) \geq x^*(-\bar{x},\bar{y})$ or $P(\bar{x},\bar{y}) \leq x^*(\bar{x},\bar{y})$. Therefore $P(\bar{x},\bar{y}) = x^*(\bar{x},\bar{y})$. As $y = -y$ and $x = -\bar{x}$ from (2) it also implies, that $P(\bar{x}+x,\bar{y}) \leq x^*(x,\bar{y}) + x^*(\bar{x},\bar{y})$, $P(\bar{x},\bar{y}+y) \leq x^*(\bar{x},y) + x^*(\bar{x},\bar{y})$ at $x \in X$, $y \in Y$ respectively. Hence we'll obtain $P(z,\bar{y}) \leq x^*(z,\bar{y})$, $P(\bar{x},\upsilon) \leq x^*(\bar{x},\upsilon)$ at $z \in X, \upsilon \in Y$. Since $-P(z,\bar{y}) \leq P(-z,\bar{y}) \leq x^*(-z,\bar{y})$, then $P(z,\bar{y}) \geq x^*(z,\bar{y})$ at $z \in X$. Therefore $P(z,\bar{y}) = x^*(z,\bar{y})$ as $z \in X$. Similarly we have, that $P(\bar{x},\upsilon) = x^*(\bar{x},\upsilon)$ at $\upsilon \in Y$. Then we'll easy obtain, that $P(z,\upsilon) \geq x^*(z,\upsilon)$ at $(z,\upsilon) \in X \times Y$. Vice versa, if $x^* \in \partial_2 P, P(z,\bar{y}) = x^*(z,\bar{y}), P(\bar{x},\upsilon) = x^*(\bar{x},\upsilon)$ at $x \in X, \upsilon \in Y$, then

$$P(\bar{x}+x,\bar{y}+y) - P(\bar{x}+x,\bar{y}) - P(\bar{x},\bar{y}+y) + P(\bar{x},\bar{y}) \geq x^*(\bar{x}+x,\bar{y}+y) -$$
$$-x^*(\bar{x}+x,\bar{y}) - x^*(\bar{x},\bar{y}+y) + x^*(\bar{x},\bar{y}) = x^*(\bar{x},\bar{y}) + x^*(\bar{x},y) +$$
$$+x^*(x,\bar{y}) + x^*(x,y) - x^*(\bar{x},\bar{y}) - x^*(x,\bar{y}) - x^*(\bar{x},\bar{y}) -$$
$$-x^*(\bar{x},y) + x^*(\bar{x},\bar{y}) = x^*(x,y).$$

The lemma is proved.

Corollary 4. *Let* $P(x,y) = P_1(x) P_2(y)$ *be a bisublinear function,* $P_1(\bar{x}) P_2(\bar{y}) \neq 0$ *and there exist* $x^* \in B(X \times Y, R)$ *such, that (2) is satisfied. Then*

$$P_1(x) = \frac{1}{P_2(\bar{y})} x^*(x,\bar{y}), \quad P_2(y) = \frac{1}{P_1(\bar{x})} x^*(\bar{x},y).$$

Let's note, that the results, obtained for biconvex operators it is possible to transfer for n convex operators. Besides, the results, obtained for $\partial_2^{\alpha} P$ it is possible to transfer for $\partial_2 P$ and vice versa.

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