Gamza F. ALIEV

THE APPROXIMATION AND CONVERGENCE OF BOUNDARY CONDITIONS OF A FOURTH ORDER EQUATION

Abstract

The straight line method is used to study the considered problems. After approximating by means of orthogonal transformation, the obtained system is reduced to canonical form. Then, applying some scheme we succeed to get approximate solution and its convergence.

There are a great number of publications on the approximation of second order equations (parabolic,hyperbolic and elliptic).The convergence rate of approximate solution to initial one for the boundary value problems of the second order equation was defined by me with the help of straight lines method and the method of quasilinearization in the paper [2]. The straight line method first was used by Rothe in the paper [1] to find the existence of solution of parabolic type equation. As for the equations of the higher orders the question about such method of substitution of derivatives by difference ratios have been studied not enough. The papers of A.Langenbach [3] where he used the method of straight lines for the approximate solution of biharmonic equation in case of trapezodial field were dedicated to these questions, but the questions about estimation of error and the convergence were not considered.

This paper is dedicated to the finding approximate solution of some boundary value problems for elliptic equations of fourth order met in the theory of flexion of thin, elastic plates, related to the most important problems of elasticity theory by the method of straight lines. Their urgency is stipulated by that multiple practical applications, where the plates as elements of the building constructions were found both in the building practice and in the machine building, in the ship building, in the aircraft building and etc. In some aspects the approximate solutions of approximation of the boundary value problems for the fourth order equations has been studied not enough.

The following problem:

$$\Delta^2 u(x,y) + P\Delta u(x,y) + qu(x,y) = f(x,y), \tag{1}$$

$$u(x,y)|_\Gamma = \varphi(x,y), \tag{2}$$

$$\left[\lambda'\Delta u + \lambda^2 \frac{\partial u}{\partial n}\right]_\Gamma = \psi(x,y) \tag{3}$$

is studied in the rectangular field D with the boundary Γ, for the definition of approximate solution and convergence, where

$$D\{x_1 \le x \le x_2, \quad y_0 \le y \le y_0 + l = Y_{N+1}\}$$

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad \Delta^2 u = \Delta(\Delta u),$$

$\dfrac{\partial u}{\partial n}$ is the derivative in the direction of the exterior normal, $p, q, \lambda^1, \lambda^2$ are constant numbers, the functions $f(x,y), \varphi(x,y)$ and $\psi(x,y)$ have on the both arguments continuos derivatives till the second order inclusively. The aim of this paper is to get the differential equations of approximate systems by the method of straight lines. For this purpose, let's draw the straight lines $y = y_k = kh \left(k = \overline{1,n};\ h = \dfrac{l}{n+1}\right)$ in the rectangular field D with the boundary Γ and let's take $y = y_k$ in (1) and let's draw the straight lines $y = y_k = kh \left(k = \overline{1,n};\ h = \dfrac{l}{n+1}\right)$ in the following way

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=y_k} = h^{-2}\left[u_{k+1}(x) - 2u_k(x) + u_{k-1}(x)\right] - \frac{h^2}{12} u_{y^4}^{(IV)}(x, \tilde{y}_k),$$

$$(y_k - h < \tilde{y}_k < y_k + h)\ ,$$

$$\left.\frac{\partial^2 u}{\partial y^4}\right|_{y=y_k} = h^{-4}\left[u_{k+2}(x) - 4u_{k+1}(x) + 6u_k(x) - 4u_{k-1}(x) + u_{k-2}(x)\right]$$

$$-\frac{h^2}{6} u_{y^6}^{(IV)}\left(x, \approx{y}_k\right), \quad \left(y_k - h < \approx{y}_k < y_k + h\right)\ ,$$

here $u_k(x) = u(x, y_k)\,, \ \left(k = \overline{1,n}\right)$; where $u_k(x)$ are discrete values of solutions of the given problem on straight lines $y = y_k$. Now applying the scheme as in the paper [1] equation (1) is approximated in the following way:

$$u_n^{(IV)}(x) + 2h^{-2}\left[u''_{n+1}(x) - 2u''_n(x) + u''_{n-1}(x)\right] +$$

$$+h^{-4}\left[u_{n+2}(x) - 4u_{n+1}(x) + 6u_n(x) - 4u_{n-1}(x) + u_{n-2}(x)\right] +$$

$$+p\left\{u''_n(x) + h^{-2}\left[u_{n+1}(x) - 2u_n(x) + u_{n-1}(x)\right]\right\} + qu_{(n)}(x) = f_n(x) + R_n(x)\,. \quad (4)$$

Then, applying method as in [4], we'll get the following system:

$$U_n^{(IV)}(x) + 2h^{-2}\left[U''_{n+1}(x) - 2U''_n(x) + U''_{n-1}(x)\right] + pU''_n(x) +$$

$$+ph^{-2}\left[U_{n+1}(x) - 2U_n(x) + U_{n-1}(x)\right] +$$

$$+h^{-4}\left[U_{n+2}(x) - 4u_{n+1}(x) + 6U_n(x) - 4U_{n-1}(x) + U_{n-2}(x)\right]$$

$$+ qU_n(x) = f_n(x)\,, \quad \left(n = \overline{1,N}\right). \quad (5)$$

Here $U_k(x)$ are the approximate solutions of problem (1)-(3) on the straight lines $y = y_k,\ k = \overline{1,n}$.

This system is a part of approximation of problem (1)-(3). Then, let's approximate the boundary conditions (3). We should note that the functions $u_{-1}(x)$, $u_{n+2}(x)$ will be included to the system (4), and respectively the functions $U_{-1}(x)$, $U_{n+2}(x)$ will be included to the system (5) at $k = \overline{1,n}$, which aren't defined out of the field D.

Let's continue the solution of the problem (1)-(3) continuously to the band h in the following way:

$$u_{-1}(x) = U_{-1}(x) = \varphi(x, y_0 - h)\,, \quad u_{n+2}(x) = U_{n+2}(x) = \varphi(x, y_{n+1} + h)\ .$$

And on the straight lines $y = y_i \quad (i = 0, n+1)$ we shall consider that the approximate solutions $U_0(x)$ and $U_{n+1}(x)$ coincide with exact solutions, i.e. $U_0(x) = u_0(x) = \varphi(x, y_0) = \varphi_0(x)$, $U_{n+1}(x) = u_{n+1}(x) = \varphi(x, y_{n+1}) = \varphi_{n+1}(x)$.

Then, the derivative by the exterior normal on the boundaries $x = x_i \ (i = 1, 2)$ coincides to within the sigh with the derivative with respect to x and on the boundaries $y = y_i \quad \left(i = \overline{0, n+1}\right)$ with the derivative with respect to y, i.e.

$$\left.\frac{\partial u}{\partial n}\right|_{x=x_1} = -\frac{\partial u}{\partial x}, \quad \left.\frac{\partial u}{\partial n}\right|_{x=x_2} = \frac{\partial u}{\partial x}, \quad \left.\frac{\partial u}{\partial n}\right|_{y=y_0} = -\frac{\partial u}{\partial y}, \quad \left.\frac{\partial u}{\partial n}\right|_{y=y_{n+1}} = \frac{\partial u}{\partial y}.$$

For approximating the boundary conditions of initial value problem, the derivatives with respect to y on straight lines $y = y_i \quad \left(i = \overline{0, n+1}\right)$ are represented by the central difference relations. Then we'll get:

$$\left.\begin{array}{r} \left\{\lambda^1 u_k''(x) + \frac{\lambda^1}{h^2}\left[u_{k+1}(x) - 2u_k(x) + u_{k-1}(x)\right] + \right. \\ \left. + \frac{\lambda^2}{2h}\left[u_1(x) - u_{-1}(x)\right]\right\}_{y=y_0} = \psi_0(x) + r_0(x) \\ \\ \left\{\lambda^1 u_k''(x) + \frac{\lambda^1}{h^2}\left[u_{k+1}(x) - 2u_k(x) + u_{k-1}(x)\right] + \right. \\ \left. + \frac{\lambda^2}{2h}\left[u_{n+2}(x) - u_n(x)\right]\right\}_{y=y_{n+1}} = \psi_{n+1}(x) + r_{n+1}(x) \end{array}\right\} \tag{6}$$

Neglecting $r_0(x)$ and $r_{n+1}(x)$ in the system (6) we'll get

$$\left.\begin{array}{r} \lambda^1 U_0''(x) + \frac{\lambda^1}{h^2}\left[U_1(x) - 2U_0(x) + U_{-1}(x)\right] + \\ + \frac{\lambda^2}{2h}\left[U_1(x) - U_{-1}(x)\right] = \psi_0(x) \\ \\ \lambda^1 U_{n+1}''(x) + \frac{\lambda^1}{h^2}\left[U_{n+1}(x) - 2U_{n+1}(x) + U_n(x)\right] + \\ + \frac{\lambda^2}{2h}\left[U_{n+2}(x) - U_n(x)\right] = \psi_{n+1}(x) \end{array}\right\} \tag{7}$$

Hence, we'll get

$$|r_0(x)| \le \frac{h^2}{6}\left(\frac{\lambda^1}{2}M_4 + \lambda^2 M_3\right),$$

$$|r_{n+1}(x)| \le \frac{h^2}{6}\left(\frac{\lambda^1}{2}M_4 + K_2 M_3\right),$$

Here M_4, M_3, K_2 are the constants, i.e. substituting the system (6) with the system (7) we make mistake of order h^2 with respect to the step h. Substituting from the system (7) to the system (5) instead of $u_0(x)$, $U_0(x)$, $U_{n+1}(x)$, $U_{-1}(x) + U_1(x)$ and $U_n(x) + U_{n+1}(x)$ the corresponding them expressions by the method of the paper (4) we'll get the following system of ordinary differential equations for definition of the approximate solutions of $U_k(x)$:

$$U_n^{(IV)}(x) + 2h^{-2}\left[U_{n-1}''(x) - 2U_n''(x)\right] + pU_n''(x) + ph^{-2}\left[U_{n-1}(x) - 2U_n(x)\right] + \\ + h^{-4}\left[U_{n-2}(x) - 4U_{n-1}(x) + 5U_n(x)\right] = F_n(x), \tag{8}$$

where

$$F_n(x) = f_n(x) + \frac{1}{(2k_1 + k_2)h^4}\{4k_1\varphi_{n+1} - \left[2k_1\varphi(x, y_{n+1} + h) - 4k_2\varphi_{n+1}(x)\right]h -$$

$$-\left[2\psi_{n+1}-6k_1\varphi''_{n+1}(x)-2pk_1\varphi_{n+1}(x)\right]h^2+\left[2k_2\varphi''_{n+1}(x)+pk_2\varphi_{n+1}(x)\right]h^3\},$$

with the boundary conditions:

$$U_k(x_i)=\varphi_k(x_i)\ ,$$
$$kU''_k(x_i)+(-1)^i k_2U'_k(x_i)=\psi_k(x_i)-k_1\varphi''_{y^2}(x_i,y_k)\ ,\qquad \left(i=1,2;\ \ k=\overline{1,n}\right)\ . \quad (9)$$

If we introduce into consederation the following vectors:

$$U(x)=[U_k(x)]_{k=1}^n\,,\quad F(x)=[F_k(x)]_{k=1}^n\,,\ \Phi(x)=[\varphi_k(x)]_{k=1}^n\ ,\ \psi(x)=[\psi_k(x)]_{k=1}^n\ ,$$

and the matrix M

$$M=\left\|\begin{array}{cccccccccc} -2 & 1 & 0 & . & . & . & 0 & 0 & 0\\ 1 & -2 & 1 & . & . & . & 0 & 0 & 0\\ 0 & 1 & -2 & . & . & . & 0 & 0 & 0\\ . & . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & . & . & . & 0 & -2 & 1\\ 0 & 0 & 0 & . & . & . & 0 & 1 & -2 \end{array}\right\|$$

then we can rewrite system (5) in the form of one matrix equation.

$$U^{IV}(x)+\left[PE+\frac{2}{h^2}M+\frac{1}{h^4}M^2\right]U(x)=F(x)\,, \qquad (10)$$

with the following boundary conditions:

$$\begin{array}{c} U(x_i)=\Phi(x_i)\ ,\\ \lambda^1U''(x_i)+(-1)^i\lambda^2U'(x_i)=\psi(x_i)-\lambda^1\Phi''(x_i)\ , \end{array} \qquad (11)$$

where $\Phi''(x_i)=\left[\varphi''_{y^2}(x,y_k)\right]_{k=1}^n$, E is a unit matrix.

Equations (10) is said to be the approximate matrix equation of the straight lines method for the stated problem.

Let's transform the system of equations of the straight line method (5) to such a system whose each equation contains only one unknown function, and let's call this system the canonical form. To this end let's lead the matrix M to the diagonal form. Let B the orthogonal transformation leading M to the diagonal form, i.e.

$$B^{-1}MB=\lambda=\left\|\begin{array}{ccccccccc} \lambda_1 & 0 & 0 & . & . & . & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & . & . & . & 0 & 0 & 0\\ . & . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & . & . & . & 0 & \lambda_{n-1} & 0\\ 0 & 0 & 0 & . & . & . & 0 & 0 & \lambda_n \end{array}\right\|$$

Then the elements of matrix λ and B have the form

$$\lambda=\|\lambda_s\|=\left\|-2\left(1+\cos\frac{\pi s}{n+1}\right)\right\|\ ,$$

$$B = B^{-1} = \|b_{ks}\| = \left\| (-1)^{k+s} \sqrt{\frac{2}{n+1}} \sin \frac{\pi ks}{n+1} \right\| \qquad \left(k, s = \overline{1,n}\ \right) \ .$$

Let's denote by $\tilde{U}(x) = BU(x)$ for any vector $U(x)$. Then applying the transformation B to the both parts of equation (10) we'll get:

$$\tilde{U}^{IV}(x) + \left[PE + \frac{2}{h^2}\lambda \right] \tilde{U}''(x) + \left[qE + \frac{P}{h^2}\lambda + \frac{1}{h^4}\lambda^2 \right] \tilde{U}(x) = \tilde{F}(x), \tag{12}$$

with the following boundary conditions:

$$\tilde{U}(x_i) = \tilde{\Phi}(x_i),$$

$$\lambda^1 \tilde{U}''(x_i) + (-1)^i \lambda^2 \tilde{U}'(x_i) = \tilde{\psi}(x_i) - \lambda^1 \tilde{\Phi}''(x_i) \ , \quad (i = 1, 2) \ . \tag{13}$$

The last equation (12) is equivalent to n unknown equations

$$\tilde{U}_s^{IV}(x) + \alpha_s^2 \tilde{U}_s''(x) + \beta_s^2 \tilde{U}_s(x) = \tilde{F}_s(x) \tag{14}$$

and the boundary conditions (13) will be equivalent to the conditions:

$$\begin{gathered} \bar{U}_s(x_i) = \bar{\varphi}_s(x_i) \ , \\ \lambda^1 \overline{U_s''}(x_i) + (-1)^i \lambda^2 \overline{U_s'}(x_i) = \bar{\psi}_s(x_i) - \lambda^1 \overline{\varphi_s''}(x_i) \ , \end{gathered} \tag{15}$$

where

$$\alpha_s^2 = -\frac{ph^2 + 2\lambda_s}{h^2} \ , \quad \beta_s^2 = \frac{\lambda_s^2 + ph^2\lambda_s + bh^4}{h^4} \ , \quad \overline{\varphi_s''}(x_i) = \overline{\varphi_{y^2}''}(x_i, y_s) \ , \quad \left(s = \overline{1,n}\right)$$

The general solution of equation (14) has the form

$$\bar{U}_s(x) = \sum_{k=1}^{n} C_{sv} e^{k_{sv}x} + \bar{U}_s^*(x) \ ,$$

where k_{sv} are the roots of the characteristic equation

$$p_s(k) = k_s^4 - \alpha_s^2 k_s^2 + \beta_s^2 = 0 \ ,$$

and $\overline{U}_s^*(x)$ is the particular solution of equation (14) that can be represented in the form

$$\overline{U}_s^*(x) = \sum_{v=1}^{4} \frac{e^{k_{sv}x}}{p_s'(k_{sv})} \int_0^x \tilde{F}_s(t) e^{-k_{sv}t} dt \qquad \left(s = \overline{1,n}\ \right).$$

Then we'll get a general solution of system (5) in the form

$$U_k(x) = \sum_{s=1}^{n} q_{ks} \overline{U}_s(x) = \sum_{s=1}^{n} q_{ks} \overline{U}_s(x) \left[\sum_{v=1}^{4} C_{sv} e^{k_{sv}x} + \tilde{U}_s^*(x) \right] \ .$$

Entering in the given solution $4n$ arbitrary derivative constants are defined from the algebraic system of linear equations obtained from (15).

So, we proved the following theorem.

Theorem. *Let there exist the solutions of the problem (1)-(3), where* $\frac{\partial u}{\partial n}$ *is a derivative of the higher normal* p, q, λ^1 *and* λ^2 *are the constant numbers, the functions* $f(x,y)$, $\varphi(x,y)$ *and* $\psi(x,y)$, *have continuous derivatives on both arguments till the second order inclusively.*

Then we'll get the approximate solutions of the problem (1)-(3) in the form:

$$U_k(x) = \sum_{s=1}^{n} q_{ks}\overline{U}_s(x) = \sum_{s=1}^{n} q_{ks}\overline{U}_s(x)\left[\sum_{v=1}^{4} C_{sv}e^{k_{sv}x} + \tilde{U}_s^*(x)\right] .$$

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Gamza F.Aliev
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.)

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