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# ON ERGODIC CHARACTERISTICS OF THE SEMI-MARKOVIAN RANDOM WALK WITH NEGATIVE DRIFT POSITIVE JUMPS AND TWO BARRIERS

**Abstract**

*In this paper the explicit form and characteristics of the ergodic distribution function of the semi-markovian random walk process with negative drift positive jumps and two delaying barriers is obtained.*

It's known that numerous interesting problems in the fields of queuing, reliability, stock control, storage, inventory theories, mathematical finance are given by means of the semi-markovian random walk process with barriers. Many studies in this topic exist in literature [2], [3], [5].

Suppose $\{\xi_i\}_{i\geq 1}$, $\{\eta_i\}_{i\geq 1}$ are the sequences of independent, positive valued and identical distributed in each sequence random variables defined on any probability space $(\Omega, \Im, P)$.

Let's construct the following renewal process $\{T_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad Y_n = \sum_{i=1}^{n} \eta_i, \quad T_0 = Y_0 = 0 \ ,$$

random walk $\{R_n\}_{n\geq 1}$

$$R_n = Y_n - T_n,$$

and Markov chain

$$x_n = \min\left\{S, \max\left\{0, x_{n-1} - \xi_n\right\} + \eta_n\right\}, \ x_0 \in [0, S], \ n = 1, 2, ...,$$

where $S$ is a fixed positive number.

Now, we can define the desired process:

$$X(t) = \max\left\{0, x_n + T_n - t\right\} \quad if \quad T_n < t \leq T_{n+1}, \quad n = 0, 1, 2, ...$$

This process form a semi-markovian random walk process with negative drift positive jumps and two delaying barriers.

Let's mark $J_f(t) = \int_0^t f(X(u))\, du$, where $f(x)$ is a measurable and bounded function on $[0, S]$. $J_f(t)$ is called an additive functional of the process $X(t)$.

Let's mark by $\tilde{L}(\lambda, z, x)$, $L^*(\lambda, z, x)$ Laplace and Laplace-Stieltjes transformations of any distribution function $L(t, z, x)$, and assume

$$\bar{L}(t, \cdot, x) = \int_0^S L(t, z, x)\, d\pi(z), \ L_1(t) * L_2(t) = \int_0^t L_2(t - s)\, dL_1(s) \ ,$$

$$L^{*n}(t)=\int_0^t L^{*(n-1)}(t-s)\,dL(s),\ n\geq 2,\ L^{*1}(t)\equiv L(t),$$

where $L(t),\ L_1(t),\ L_2(t)$ are some distribution functions.

Suppose that the distribution functions of $\xi_i$ and $\eta_i$ are known:

$$\Phi(t)=P\{\xi_1<t\},\quad F(x)=P\{\eta_1<x\},\ x>0,$$

$$\pi(z)=P\{\chi_1<z\}=P\{\min\{S,\eta_1\}<z\}=\begin{cases}F(z), & z<S\\ 1, & z\geq S\ .\end{cases}$$

The following notation will be used in this study: are probability characteristics of the renewal processes $\{T_n\}_{n\geq 1}$, $\{Y_n\}_{n\geq 1}$, and random walk $\{R_n\}_{n\geq 1}$.

$$a_n(t,z,x)=P\left\{\eta_i<z+R_i<S,\ i=\overline{1,n};\ z+Y_n-t<x;\ T_n<t\leq T_{n+1}\right\},$$

$$b_n(t,z)=P\left\{\eta_i<z+R_i<S,\ i=\overline{1,n-1};\ z+R_n>\eta_n;\ z+R_n\geq S;\ T_n<t\right\},$$

$$a_0(t,z,x)=P\{z-x\leq t<\xi_1\},\ b_0(t,z)=0,$$

$$A(t,z,x)=\sum_{n=0}^{\infty}a_n(t,z,x),\quad B(t,z)=\sum_{n=1}^{\infty}b_n(t,z),\ U(t,S)=\sum_{n=1}^{\infty}[B(t,S)]^{*n}\ .$$

Let's mark

$$G(t,z,x)=P_z\{X(t)<x;\ \ \tau_1\geq t\}\ .$$

Let's mark characteristic function and explicit form of ergodic distribution of process $X(t)$ consequently

$$\varphi_X(\lambda)\equiv\lim_{t\to\infty}Ee^{i\lambda X(t)},\quad \lambda>0\ \ and\ \ \ Q(x)\equiv\lim_{t\to\infty}P\{X(t)<x\}\ .$$

Define the first falling moment to zero state

$$\gamma_1=\inf\{t>0:X(t)=0\}$$

and the first exit moment from the zero state

$$\tau_1=\inf\{t>\gamma_1:X(t)>0\}$$

of process $X(t)$. Suppose $\inf\{\varnothing\}=\infty$.

$\gamma_1$ and $\tau_1$ are called boundary functionals of the process $X(t)$.

Let the following conditions be fulfilled:

i) $E\xi_1=\int_0^\infty td\Phi(t)<\infty$ ,

ii) $P\{\eta_1>\xi_1\}>0,\quad P\{\eta_1<\xi_1\}>0$ ,

iii) $\eta_1$ is non-arithmetic random variable.

In [7] it is proved that the conditions i)-iii) are satisfied, then the process $X(t)$ is ergodic i.e. there exists a $\lim\limits_{t\to\infty}\dfrac{J_f(t)}{t}$ and it's not random with probability 1.

The main result of this paper is presented in the next theorem, where the ergodic distribution of the process $X(t)$ can be expressed by means of renewal processes $\{T_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ :

**Theorem.** *Suppose the conditions i)-iii) are satisfied. Then the following relation holds with probability 1:*

$$S_f = \lim_{t\to\infty} \frac{J_f(t)}{t} = \frac{1}{E\tau_1}\left[\tilde{\tilde{A}}_f(0,\cdot,*) + B^*(0,\cdot)\left[1 - B^*(0,S)\right]^{-1} \cdot \tilde{A}_f(0,S,*)\right] ,$$

*where*

$$E\tau_1 = \tilde{\tilde{A}}_f(0,\cdot,S) + \bar{B}^*(0,\cdot)\left[1 - B^*(0,S)\right]^{-1} \cdot \tilde{A}(0,S,S) \ .$$

**Proof.** Accordingly Skorohod's theorem ([1] p.362), for any measurable and bounded function $f(x)$ on $[0,S]$ it holds:

$$S_f = \lim_{t\to\infty} \frac{J_f(t)}{t} = \frac{1}{E\tau_1} \int_0^\infty \int_0^S \int_0^S f(x) P_z\{\tau_1 \geq t,\ X(t) \in dx\} \, d\pi(z) \, dt \ . \tag{1}$$

As is known from [6], one dimensional distribution function of process $X(t)$ has the following form:

$$G(t,z,x) = A(t,z,x) + B(t,z) * U_B(t,S) * A(t,S,x) , \tag{2}$$

where $A(t,z,x)$, $B(t,z)$ and $U_B(t,S)$ are probability characteristics of the renewal processes $\{T_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$.

Applying the Laplace transform to $t$, we obtain from (2):

$$\tilde{G}(\lambda,z,x) = \tilde{A}(\lambda,z,x) + B^*(\lambda,z) \cdot U_B^*(\lambda,S) \cdot \tilde{A}(\lambda,S,x) \ . \tag{3}$$

In [5] it is proved that $E\tau_1 < \infty$ . Then the $\lim_{\lambda\to 0} \tilde{G}(\lambda,z,x)$ exists and finite. Taking the limit as in $\lambda \to 0$ (3), we have

$$\tilde{G}(0,z,x) = \tilde{A}(0,z,x) + B^*(0,z) \cdot U_B^*(0,S) \cdot \tilde{A}(0,S,x) \tag{4}$$

and

$$\tilde{\tilde{G}}(0,\cdot,x) = \tilde{\tilde{A}}(0,\cdot,x) + \bar{B}^*(0,\cdot) \cdot U_B^*(0,S) \cdot \tilde{A}(0,S,x) \ . \tag{5}$$

Substituting (5) into (1), we get:

$$S_f = \lim_{t\to\infty} \frac{J_f(t)}{t} = \frac{1}{E\tau_1}\left[\tilde{\tilde{A}}_f(0,\cdot,*) + \bar{B}^*(0,\cdot)\left[1 - B^*(0,S)\right]^{-1} \cdot \tilde{A}_f(0,S,*)\right] ,$$

where

$$\tilde{\tilde{A}}_f(0,\cdot,*) = \int_0^S f(x) \, d_x\tilde{A}(0,V,x) , \ \tilde{A}_f(0,S,*) = \int_0^S f(x) \, d_x\tilde{A}(0,S,x) \ ,$$

That's known from [5]

$$E\tau_1 = \tilde{\tilde{A}}_f(0,\cdot,S) + \bar{B}^*(0,\cdot)\left[1 - B^*(0,S)\right]^{-1} \cdot \tilde{A}(0,S,S) \ .$$

This completes the proof.

**Corollary 1.** Under conditions of the Theorem the explicit form of ergodic distribution function of process $X(t)$ is given as:

$$Q(x) = \frac{1}{E\tau_1}\left[\tilde{\bar{A}}(0,\cdot,x) + \bar{B}^*(0,\cdot)\left[1 - B^*(0,S)\right]^{-1}\cdot \tilde{A}_f(0,S,x)\right], \quad x \in [0,S] \; .$$

**Corollary 2.** Under conditions of the Theorem the characteristic function of ergodic distribution of process $X(t)$ has the following form:

$$\varphi_X(\lambda) = \frac{1}{E\tau_1}\left[\int_0^S e^{i\lambda x} d\tilde{\bar{A}}(0,\cdot,x) + \bar{B}^*(0,\cdot)\left[1 - B^*(0,S)\right]^{-1}\cdot \int_0^S e^{i\lambda x} d\ \tilde{A}_f(0,S,x)\right],$$

where $\lambda > 0$.

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