APPLIED PROBLEMS OF MATHEMATICS AND MECHANICS

Turab M. AKHMEDOV

THE ROLE OF FRACTIONAL OPERATORS IN ELECTRODYNAMICS

Abstract

In the present paper the fractional operator rot^2 is investigated as a fractional operator of the known rotor operator rot and some applications and physical interpretation of these operators in some problems of electrodynamiics are considered.

Introduction. At the present time the fractionalization of transformations and operators is of great interest both from theoretical and practical point of view. For example, the fractional Fourier transformation and its applications were studied by several authors: Namias, Lohmann, Mendlovic and Ozaktas, Shamir and Cohen [2-4], and fractionalization of Hankel transformation was considered in [5]. Having the operator L it is possible to consider fractional operator, which is denoted by L^α, where the parameter α takes real values or even can be complex. The new operator L^α expands the initial operator L, in the sense, that the operator L is obtained as a particular case of fractionalized operator at some value of α (ordinary at $\alpha = 1,\ L^\alpha|_{\alpha=1} = L$).

In the given work the fractional operator rot^α is investigated as fractional operator of the known rotor operator *rot* and some applications and physical interpretation of these operators in some electrodynamics problems are considered.

The operator *rot* is one of the key operators of the theory of electromagnetic field. In particular, in the simplest case electric and magnetic fields are connected among themselves with the help of rotor in the Maxwell equations in differential form [6] in lack of the sources:

$$\vec{E} = -\frac{1}{ik} rot\left(\eta\vec{H}\right), \quad \eta\vec{H} = \frac{1}{ik} rot\vec{E} \tag{1}$$

where $k = \dfrac{2\pi}{\lambda}$ is a wave number, $\eta = \sqrt{\mu/\varepsilon}$ is an impedance, ε, μ are dielectrical and magnetic penetrabilities.

The new fractional operator rot^α, introduced here, where $0 \le \alpha \le 1$, has the following properties: at $\alpha = 1$ we obtain the ordinary operator $rot^\alpha|_{\alpha=1} = rot$, and at $\alpha = 0$ we obtain a unit operator $rot^\alpha|_{\alpha=1} = I$.

First the fructionalized (or fractional) rotor was introduced by N.Engheta [1], where fractional rotor was used for getting new "fractional" solutions of Maxwell equations, as a result of application rot^α to the fixed solution $\left(\vec{E}, \eta\vec{H}\right)$:

$$\vec{E}^\alpha = \frac{1}{(ik)^\alpha} rot^\alpha \vec{E}, \quad \eta\vec{H}^\alpha = \frac{1}{(ik)^\alpha} rot^\alpha\left(\eta\vec{H}\right). \tag{2}$$

The "fractional" field $\left(\vec{E}^{\alpha}, \eta\vec{H}^{\alpha}\right)$, obtained like that, is characterized by the fractional order α and defines:

(1) the initial field $\left(\vec{E}, \eta\vec{H}\right)$ at $\alpha = 0$;

(2) the dual field [7] $\left(\vec{E}^{1}, \eta\vec{H}^{1}\right) = \left(\eta\vec{H}^{0}, -\vec{E}^{0}\right)$ at $\alpha = 1$.

As $0 < \alpha < 1$ the fractional field describes intermediate solution between initial and dual solutions.

The obtained result can be interpreted as generalization of the known duality principle for Maxwell equation [6, 7]. Namely: having the solution $\left(\vec{E}, \eta\vec{H}\right)$, satisfying the Maxwell equations with the medium parameters ε, μ and applying the fractional rotor to this solution, we obtain a new field, again satisfying the Maxwell equations with the same parameters ε, μ. As a particular case, at $\alpha = 1$ the fractional field describes the dual field, taking part in formulation of the known duality principle.

In the paper [1] the general scheme of obtaining fractionalized operator from arbitrary linear operator is given and the expression for rot^{α} function of one variable $\vec{P} = \vec{P}(z) = P_x(z)\vec{x} + P_y(z)\vec{y} + P_z(z)\vec{z}$ is represented in the explicit form:

$$rot^{\alpha}\vec{P}(z) = \left[\cos\left(\frac{\alpha\pi}{2}\right)_{-\infty} D_z^{\alpha} P_x(z) - \sin\left(\frac{\alpha\pi}{2}\right)_{-\infty} D_z^{\alpha} P_y(z)\right]\vec{x} +$$

$$+\left[\sin\left(\frac{\alpha\pi}{2}\right)_{-\infty} D_z^{\alpha} P_x(z) + \cos\left(\frac{\alpha\pi}{2}\right)_{-\infty} D_z^{\alpha} P_y(z)\right]\vec{y} + \delta_{0\alpha -\infty} D_z^{\alpha} P_z(z)\vec{z} \qquad (3)$$

where $\delta_{0\alpha}$ is Kronecker's symbol, the operator ${}_{-\infty}D_z^{\alpha}$ means Riemann-Liouville fractional integral [10], which is defined as

$${}_{-\infty}D_z^{\alpha}P(z) = \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{z}\frac{P(u)\,du}{(z-u)^{\alpha+1}}, \quad 0 < \alpha < 1$$

where $\Gamma(x)$ is Euler gamma-function.

However for many scattering and radiation problems definition (3) is not enough: it is necessary to have the expression for rot^{α} for functions, dependent on two or three variables. In the general case for arbitrary function it is sufficiently difficult to obtain in the explicit form the expression for a fractional rotor.

The aim of the given work is to obtain the expression for fractional rotor for the function of two or three variables, expressed by exponential functions.

As it will be shown below, for a function of three variables, expressed by exponents for the operator rot^{α} $(0 \le \alpha \le 1)$ the following representation holds:

$$rot^{\alpha}\left[\vec{z}e^{iax+iby+icz}\right] = \frac{1}{k^2}\left(i^{\alpha}\delta_{0\alpha}ac + (ik)^{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)kb - (ik)^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)ac\right)F\vec{x} +$$

$$+\frac{1}{k^2}\left(i^{\alpha}\delta_{0\alpha}bc - (ik)^{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)ka - (ik)^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)bc\right)F\vec{y} +$$

$$+\frac{1}{k^2}\left(i^{\alpha}\delta_{0\alpha}c^2 + (ik)^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\left(a^2+b^2\right)\right)F\vec{z}. \qquad (4)$$

The consideration of the exponents is caused by that they play an important role in electrodynamics. For example, plane waves are described with the help of exponents. Moreover, there are decompositions of cylindrical and spherical waves by the plane waves [9].

The obtained representations of the fractional rot^{α} later can be used in the concrete scattering and radiation problems for obtaining "intermediate solutions", applying the fractional rotor to some fixed solution of the problem.

At that the parameter α, characterizing the fractionalization of solution, will be defined by the various values of physical quantities, describing the problem.

1. The fractional operator from a linear operator. We can consider the operator *rot* as a linear operator. Therefore, let's consider the general scheme of getting fractional operator from the linear operator, introduced in [1]. For this reason, consider the class of linear operators $\{L, L : C^n \to C^n\}$, where C^n is a space of n-dimensional vectors over the field of complex numbers.

The operator L^{α} is called fractionalized (fractional) operator from the linear operator L, if the following conditions are fulfilled:

1. as $\alpha = 1$ $\; L^{\alpha}|_{\alpha=1} = L$;
2. as $\alpha = 0$ $\; L^{\alpha}|_{\alpha=0} = I$, where I is an identity (unit) operator
3. semigroup properties $L^{\alpha}L^{\beta} = L^{\beta}L^{\alpha} = L^{\alpha+\beta}$.

Let the linear operator L have the eigen vectors $\left\{\vec{A}_m, \; m = 1..n\right\}$ and the corresponding eigen values $\{a_m, m = 1..n\}$. Then

$$L\left(\vec{A}_m\right) = a_m \vec{A}_m \; .$$

$\left\{\vec{A}_m\right\}$ form full linearly independent system of eigen vectors in the space C^n. It means, that the arbitrary vector $\vec{H}$ and C^n can be represented in the form of linear combinations of these vectors

$$\vec{H} = \sum_{m=1}^{n} c_m \vec{A}_m$$

with some decomposition coefficients c_m.

Define the fractional operator L^{α}, as operator with the same eigen vectors $\left\{\vec{A}_m, \; m = 1..n\right\}$, but with eigen values $\{(a_m)^{\alpha}, \; m = 1..n\}$, i.e.

$$L^{\alpha}\left(\vec{A}_m\right) = (a_m)^{\alpha} \vec{A}_m \; .$$

Let's remark, that the choice of branch of multivalued quantity $(a_m)^{\alpha}$ each time is carried out proceeding from physical conditions of the concrete problem.

So, action of the operator L^{α} on the arbitrary vector $\vec{H}$ from C^n will be written in the from:

$$L^{\alpha}\left(\vec{H}\right) = L^{\alpha}\left(\sum_{m=1}^{n} c_m \vec{A}_m\right) = \sum_{m=1}^{n} c_m L^{\alpha}\left(\vec{A}_m\right) = \sum_{m=1}^{n} (a_m)^{\alpha} c_m \vec{A}_m \; . \qquad (5)$$

This correlation defines the fractional operator L^{α} by eigen values and eigen vectors of the operator L. It is easy to see, that this definition satisfies all three conditions in definition of fractionalized operator.

2. Fractional rotor rot^{α}.

2.1. The general case. It is known, that for the three-dimensional vector $\vec{P}$ as the function of variables x, y, z in the Cartesian system of coordinates:

$$\vec{P}(x,y,z) = P_x(x,y,z)\,\vec{x} + P_y(x,y,z)\,\vec{y} + P_z(x,y,z)\,\vec{z}$$

the rotor operator has the form:

$$rot\vec{P} = \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right)\vec{x} + \left(\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right)\vec{y} + \left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right)\vec{z}$$

where $\vec{x}, \vec{y}, \vec{z}$ are unit vectors of Cartesian system of coordinates.

Apply the Fourier transformation from the space x, y, z to the space $\vec{k}(k_x, k_y, k_z)$ to the vectors $\vec{P}$ and $rot\vec{P}$, assuming that the Fourier transformations $\overset{\wedge}{\vec{P}}(k_x, k_y, k_z) \equiv F_k\left(\vec{P}(x,y,z)\right)$ and $\overset{\wedge}{\vec{R}}(k_x, k_y, k_z) \equiv F_k\left(rot\vec{P}(x,y,z)\right)$ exist. This is fulfilled, for example, if we'll necessitate, that $P'_x, P'_y, P'_z \in L_2(-\infty, \infty)$ is a class of functions with summable square [12].

$$\overset{\wedge}{\vec{P}} \equiv F_k\left(\vec{P}(x,y,z)\right) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{P}(x,y,z)\, e^{-ik_x x - ik_y y - ik_z z} dxdydz\ .$$

The Fourier transformation $\overset{\wedge}{\vec{R}} \equiv F_k\left(rot\vec{P}(x,y,z)\right)$ of the rotor $\vec{P}$ we can represent in the following form:

$$\overset{\wedge}{\vec{R}} \equiv F_k\left(rot\vec{P}(x,y,z)\right) = i\vec{k} \times F_k\left(\vec{P}(x,y,z)\right) = i\vec{k} \times \overset{\wedge}{\vec{P}}. \tag{6}$$

So, in the space of $\vec{k}$ images of Fourier a rotor operator can be represented as a vector product of the vector $i\vec{k}$ on the vector $\overset{\wedge}{\vec{P}}$. Hence, for defining the fractional rotor rot^{α}, it is necessary to define fractional operator from the vector product $i\vec{k} \times \overset{\wedge}{\vec{P}}$ in the space $\vec{k}$.

The operator $L = \left(i\vec{k}\times\right)$ at the fixed vector $\vec{k}$ is a linear operator, acting on the arbitrary vector $\vec{H}$ as $L\left(\vec{H}\right) = i\vec{k} \times \vec{H}$. So, following the described scheme of getting a fractionalized operator, it is possible to define the fractional operator $L^{\alpha} = \left(i\vec{k}\times\right)^{\alpha}$:

$$L^{\alpha}\overset{\wedge}{\vec{P}} = \left(i\vec{k}\times\right)^{\alpha}\overset{\wedge}{\vec{P}}\ .$$

Applying the inverse Fourier transformation (passing inversely to the space x, y, z), we obtain a fractional rotor from the vector $\vec{P}$:

$$rot^{\alpha}\vec{P} = F_k^{-1}\left(L^{\alpha}\overset{\wedge}{\vec{P}}\right) .$$

It is easy to see, that the operator $L_0 = \left(\vec{k}\times\right)$ has the following eigen values and eigen vectors:

$$\vec{A}_1 = \vec{k}\left(k_x, k_y, k_z\right), \quad a_1 = 0 \;;$$

$$\vec{A}_2\left(ikk_z - k_y k_x, k_x^2 + k_z^2, -ikk_x - k_y k_z\right), \; a_2 = ik$$

$$\vec{A}_3\left(-ikk_z - k_y k_x, k_x^2 + k_z^2, ikk_x - k_y k_z\right), \; a_3 = -ik$$

where $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

The arbitrary vector $\vec{H} = H_x\vec{x}_k + H_y\vec{y}_k + H_z\vec{z}_k$ in the Cartesian system of coordinates is represented in the form of linear combination by eigen vectors:

$$\vec{H} = c_1\vec{A}_1 + c_2\vec{A}_2 + c_3\vec{A}_3,$$

where the addends $c_m\vec{A}_m$ are defined as:

$$\begin{aligned} c_1\vec{A}_1 = {} & \frac{1}{k^2}\left(k_x^2 H_x + k_x k_y H_y + k_x k_z H_z\right)\vec{x}_k + \\ & + \frac{1}{k^2}\left(k_x k_y H_x + k_y^2 H_y + k_y k_z H_z\right)\vec{y}_k + \\ & + \frac{1}{k^2}\left(k_x k_z H_x + k_y k_z H_y + k_z^2 H_z\right)\vec{z}_k \end{aligned} \tag{7}$$

$$\begin{aligned} c_2\vec{A}_2 = {} & \frac{1}{2k^2}\left[\left(k^2 - k_x^2\right)H_x + \left(ikk_z - k_x k_y\right)H_y + \left(-ikk_y - k_x k_z\right)H_z\right]\vec{x}_k + \\ & + \frac{1}{2k^2}\left[\left(-ikk_z - k_x k_y\right)H_x + \left(k^2 - k_y^2\right)H_y + \left(ikk_x - k_y k_z\right)H_z\right]\vec{y}_k + \\ & + \frac{1}{2k^2}\left[\left(ikk_y - k_x k_z\right)H_x + \left(-ikk_x - k_y k_z\right)H_y + \left(k^2 - k_z^2\right)H_z\right]\vec{z}_k \end{aligned} \tag{8}$$

$$\begin{aligned} c_3\vec{A}_3 = {} & \frac{1}{2k^2}\left[\left(k^2 - k_x^2\right)H_x + \left(-ikk_z - k_x k_y\right)H_y + \left(ikk_y - k_x k_z\right)H_z\right]\vec{x}_k + \\ & + \frac{1}{2k^2}\left[\left(ikk_z - k_x k_y\right)H_x + \left(k^2 - k_y^2\right)H_y + \left(-ikk_x - k_y k_z\right)H_z\right]\vec{y}_k + \\ & + \frac{1}{2k^2}\left[\left(-ikk_y - k_x k_z\right)H_x + \left(ikk_x - k_y k_z\right)H_y + \left(k^2 - k_z^2\right)H_z\right]\vec{z}_k \; . \end{aligned} \tag{9}$$

So the fractional operator $L^{\alpha} = \left(i\vec{k}\times\right)^{\alpha} = i^{\alpha}\left(\vec{k}\times\right)^{\alpha}$ is represented in the form:

$$L^{\alpha}\left(\overset{\wedge}{\vec{P}}\right) = i^{\alpha}\sum_{m=1}^{3}\left(a_m\right)^{\alpha} c_m\vec{A}_m. \tag{10}$$

For getting the representation for the fractional rotor $rot^{\alpha}\vec{P}$, we have to apply the inverse Fourier transformation to expression (10):

$$rot^{\alpha}\vec{P} = F_k^{-1}\left(L^{\alpha}\left(\overset{\wedge}{\vec{P}}\right)\right) =$$

$$= F_k^{-1}\left(\delta_0^{\alpha}k_x^2\hat{P}_x\vec{x}_k - \cos\left(\alpha\pi/2\right)k_x^2k^{\alpha-2}\hat{P}_x\vec{x}_k + \cos\left(\alpha\pi/2\right)k^{\alpha}\hat{P}_x\vec{x}_k + ...\right)$$

2.2. The one-variable function. The particular case, when the initial vector $\vec{P}$ depends only on one coordinate z, i.e. $\vec{P} = \vec{P}(z) = P_x(z)\vec{x} + P_y(z)\vec{y} + P_z(z)\vec{z}$ was considered in [1].

In this case the Fourier transformation

$$\overset{\wedge}{\vec{R}} \equiv F_k\left(rot\vec{P}(z)\right) = i\vec{k}\times F_k\left(\vec{P}(z)\right) = (ik_z\vec{z}_k\times)\overset{\wedge}{\vec{P}},$$

and the fractional operator

$$L^{\alpha} = \left(i\vec{k}\times\right)^{\alpha} = (ik_z)^{\alpha}(\vec{z}_k\times)^{\alpha}\ .$$

Following the described above method of getting a fractional operator, we'll find the eigen vectors and eigen values of the operator $(\vec{z}_k\times)$:

$$\vec{A}_1(0,0,1) = \vec{z}_k,\ a_1 = 0;$$

$$\vec{A}_2(i,1,0) = i\vec{x}_k + \vec{y}_k,\ a_2 = i;$$

$$\vec{A}_3(-i,1,0) = -i\vec{x}_k + \vec{y}_k,\ a_3 = -i\ .$$

The arbitrary vector $\vec{H} = H_x\vec{x}_k + H_y\vec{y}_k + H_z\vec{z}_k$ in the space $(\vec{x}_k, \vec{y}_k, \vec{z}_k)$ is represented in form of linear combinations by the eigen vectors $\vec{H} = c_1\vec{A}_1 + c_2\vec{A}_2 + c_3\vec{A}_3$, where addends are calculated by the following formulae:

$$c_1\vec{A}_1 = H_z\vec{z}_k,$$

$$c_2\vec{A}_2 = \frac{1}{2}(H_x + iH_y)\vec{x}_k + \frac{1}{2}(-H_x + H_y)\vec{y}_k, \tag{11}$$

$$c_3\vec{A}_3 = \frac{1}{2}(H_x - iH_y)\vec{x}_k + \frac{1}{2}(iH_x + H_y)\vec{y}_k.$$

So, the fractional operator

$$(\vec{z}_k\times)^{\alpha}\vec{H} = \sum_{m=1}^{3}(a_m)^{\alpha}c_m\vec{A}_m \tag{12}$$

or in detail

$$(\vec{z}_k\times)^{\alpha}\vec{H} = 0^{\alpha}H_z\vec{z}_k + i^{\alpha}\left[\frac{1}{2}(H_x + iH_y)\vec{x}_k + \frac{1}{2}(-iH_x + H_y)\vec{y}_k\right] +$$

$$+(-i)^{\alpha}\left[\frac{1}{2}(H_x-iH_y)\vec{x}_k+\frac{1}{2}(iH_x+H_y)\vec{y}_k\right].$$

In particular, if $\vec{H}=\vec{x}_k$, then $(\vec{z}_k\times)^{\alpha}\vec{x}_k=\cos\left(\frac{\alpha\pi}{2}\right)\vec{x}_k+\sin\left(\frac{\alpha\pi}{2}\right)\vec{y}_k$.

At $\alpha=1$, we obtain the known correlation for the vector product: $(\vec{z}_k\times)^{\alpha}|_{\alpha=1}\vec{x}_k=(\vec{z}_k\times)\vec{x}_k=\vec{y}_k$.

At $\alpha=0$, we have the unit operator $(\vec{z}_k\times)^{\alpha}|_{\alpha=0}\vec{x}_k=I\vec{x}_k=\vec{x}_k$.

Using the formulae for decomposition of the vector $\overset{\wedge}{\vec{P}}\left(\hat{P}_x,\hat{P}_y,\hat{P}_z\right)$ by the eigen vectors $c_m\vec{A}$, from (12) we have the representation for the fractional operator $L^{\alpha}=\left(i\vec{k}\times\right)^{\alpha}$:

$$L^{\alpha}\left(\overset{\wedge}{\vec{P}}\right)=(ik_z)^{\alpha}\sum_{m=1}^{3}(a_m)^{\alpha}c_m\vec{A}_m=$$

$$=(ik_z)^{\alpha}0^{\alpha}\hat{P}_z\vec{z}_k+\frac{1}{2}(ik_z)^{\alpha}\left(\hat{P}_y-i\hat{P}_x\right)(i\vec{x}_k+\vec{y}_k)+$$

$$+\frac{1}{2}(-ik_z)^{\alpha}\left(\hat{P}_y-i\hat{P}_x\right)(-i\vec{x}_k+\vec{y}_k)\ . \tag{13}$$

Grouping, after transformations, finally we obtain:

$$L^{\alpha}\left(\overset{\wedge}{\vec{P}}\right)=(ik_z)^{\alpha}\delta_{0\alpha}\hat{P}_z\vec{z}_k+(ik_z)^{\alpha}\left[\hat{P}_x\cos\left(\frac{\pi\alpha}{2}\right)-\hat{P}_y\sin\left(\frac{\pi\alpha}{2}\right)\right]\vec{x}_k+$$

$$+(ik_z)^{\alpha}\left[\hat{P}_x\sin\left(\frac{\pi\alpha}{2}\right)+\hat{P}_y\cos\left(\frac{\pi\alpha}{2}\right)\right]\vec{y}_k. \tag{14}$$

Using the identity $F_k^{-1}\left((ik_z)^{\alpha}\hat{U}(z)\right)={}_{-\infty}D_z^{\alpha}U(z)$, applying to equation (14) the inverse Fourier transformation, finally we obtain the following representation for the fractional operator $rot^{\alpha}\vec{P}(z)$ in the Cartesian system of coordinates (x,y,z):

$$rot^{\alpha}\vec{P}(z)=\left[\cos\left(\frac{\alpha\pi}{2}\right)_{-\infty}D_z^{\alpha}P_x(z)-\sin\left(\frac{\alpha\pi}{2}\right)_{-\infty}D_z^{\alpha}P_y(z)\right]\vec{x}+$$

$$+\left[\sin\left(\frac{\alpha\pi}{2}\right)_{-\infty}D_z^{\alpha}P_x(z)+\cos\left(\frac{\alpha\pi}{2}\right)_{-\infty}D_z^{\alpha}P_y(z)\right]\vec{y}+\delta_{0\alpha-\infty}D_z^{\alpha}P_z(z)\vec{z}. \tag{15}$$

The obtained representation (15) is a generalization of the known operator *rot* for fractional case.

It is easy to check, that

$$rot^{\alpha}\vec{P}(z)=\begin{cases} rot\vec{P}(z), & \alpha=1\\ \vec{P}(z), & \alpha=0\end{cases}$$

2.3. The fractional rotor for the exponent of two variables. We'll apply the described above scheme for particular case, when the function is represented in the form

$$\vec{E}=\vec{z}F(x,y)=\vec{z}e^{iax+iby}$$

In this case the Fourier transformation

$$\overset{\wedge}{\vec{R}} \equiv F_k\left(rot\vec{E}\right) = i\vec{k}\times F_k\left(\vec{E}\right),$$

where $F_k\left(\vec{E}\right) = \overset{\wedge}{\vec{E}}\left(0,0,\hat{F}\right)$ and $\vec{k} = \vec{k}\left(k_x,k_y,0\right)$.

Let's denote $L^\alpha = \left(i\vec{k}\times\right)^\alpha$ as a fractional operator in the space of Fourier images. The eigen-values of this operator

$$a_1 = 0, \quad a_2 = ik, \;\; a_3 = -ik\ .$$

From formulas (7)-(9) for our case $\vec{k} = \vec{k}\left(k_x,k_y,0\right)$ we obtain, that the vector $\overset{\wedge}{\vec{E}} = \vec{z}_k\hat{F}$ in the space $\left(\vec{x}_k,\vec{y}_k,\vec{z}_k\right)$ is represented in the form of linear combination by the eigen vectors $\overset{\wedge}{\vec{E}} = c_1\vec{A}_1 + c_2\vec{A}_2 + c_3\vec{A}_3$, where addends are calculated by the following formulae:

$$c_1\vec{A}_1 = 0,$$

$$c_2\vec{A}_2 = -\frac{ik_y}{2k}\hat{F}\vec{x}_k + \frac{ik_x}{2k}\hat{F}\vec{y}_k + \frac{1}{2}\hat{F}\vec{z}_k \tag{16}$$

$$c_3\vec{A}_3 = \frac{ik_y}{2k}\hat{F}\vec{x}_k - \frac{ik_x}{2k}\hat{F}\vec{y}_k + \frac{1}{2}\hat{F}\vec{z}_k\ .$$

So

$$L^\alpha\left(\overset{\wedge}{\vec{E}}\right) = i^\alpha\sum_{m=1}^{3}\left(a_m\right)^\alpha c_m\vec{A}_m \tag{17}$$

and

$$rot^\alpha\left[\vec{E}\right] = F_k^{-1}\left[L^\alpha\left(\overset{\wedge}{\vec{E}}\right)\right]\ .$$

Substituting (16) in (17) we have

$$L^\alpha\left(\overset{\wedge}{\vec{E}}\right) = i^\alpha\sum_{m=1}^{3}\left(a_m\right)^\alpha c_m\vec{A}_m =$$

$$= i^\alpha\left\{(ik)^\alpha\left[-\frac{ik_y}{2k}\hat{F}\vec{x}_k + \frac{ik_x}{2k}\hat{F}\vec{y}_k + \frac{1}{2}\hat{F}\vec{z}_k\right] + \right.$$

$$\left. + (-ik)^\alpha\left[\frac{ik_y}{2k}\hat{F}\vec{x}_k - \frac{ik_x}{2k}\hat{F}\vec{y}_k + \frac{1}{2}\hat{F}\vec{z}_k\right]\right\} = \tag{18}$$

$$= (ik)^\alpha\left[-\hat{F}\vec{x}_k\frac{ik_y}{2k}\left(i^\alpha - (-i)^\alpha\right) + \hat{F}\vec{y}_k\frac{ik_x}{2k}\left(i^\alpha - (-i)^\alpha\right) + \hat{F}\vec{z}_k\frac{1}{2}\left(i^\alpha + (-i)^\alpha\right)\right]\ .$$

Choosing in expression (18) the branches of the multivalued functions i^α so, that

$$i^\alpha - (-i)^\alpha = 2i\sin\left(\frac{\pi\alpha}{2}\right),\ i^\alpha + (-i)^\alpha = 2\cos\left(\frac{\pi\alpha}{2}\right)\ .$$

Let's note, that the choice of the multivalued branch is always connected with conditions of a concrete physical problem.

Finally we obtain:

$$L^{\alpha}\left(\overset{\wedge}{\vec{E}}\right)=(ik)^{\alpha}\left[\sin\left(\frac{\pi\alpha}{2}\right)\frac{k_y}{k}\hat{F}\vec{x}_k-\sin\left(\frac{\pi\alpha}{2}\right)\frac{k_x}{k}\hat{F}\vec{y}_k+\cos\left(\frac{\pi\alpha}{2}\right)\hat{F}\vec{z}_k\right]. \tag{19}$$

Fourier transformation of the function $F(x,y)$:

$$\hat{F}(k_x,k_y)=F_k\left(e^{iax+iby}\right)=\delta(k_x-a)\,\delta(k_y-b)\;. \tag{20}$$

Putting (20) to expression (19) for $L^{\alpha}\left(\overset{\wedge}{\vec{E}}\right)$, then applying the inverse Fourier transformation we obtain the expression for the fractional rotor in the following form:

$$rot^{\alpha}\left[\vec{z}e^{iax+iby}\right]=$$

$$=(ik)^{\alpha}e^{iax+iby}\left[\sin\left(\frac{\pi\alpha}{2}\right)\frac{b}{k}\vec{x}-\sin\left(\frac{\pi\alpha}{2}\right)\frac{a}{k}\vec{y}+\cos\left(\frac{\pi\alpha}{2}\right)\vec{z}\right], \tag{21}$$

where k is defined as $k^2=a^2+b^2$.

From (21) obviously we obtain, that as $\alpha=0$ we have the identity operator $rot^{\alpha}|_{\alpha=0}\left[\vec{E}\right]=\vec{E}$ and as $\alpha=1$ we obtain the ordinary rotor operator $rot^{\alpha}|_{\alpha=1}\left[\vec{E}\right]=rot\vec{E}$.

In particular case, when $b=0$ we have the one-variable function $\vec{E}=\vec{z}F(x)=\vec{z}e^{iax}$ and from (21) we have

$$rot^{\alpha}\left[\vec{z}e^{iax}\right]=-(ik)^{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)\frac{a}{k}e^{iax}\vec{y}+(ik)^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)e^{iax}\vec{z}$$

that is adjusted with expression (15).

2.4. The fractional rotor of exponent of three-variables. By the similar way we obtain the expression for fractional rotor for three-variables function, represented in the form:

$$\vec{E}=\vec{z}F(x,y)=\vec{z}e^{iax+iby+icz}\;.$$

As earlier

$$rot^{\alpha}\left[\vec{E}\right]=F_k^{-1}\left[L^{\alpha}\left(\overset{\wedge}{\vec{E}}\right)\right]=F_k^{-1}\left[i^{\alpha}\sum_{m=1}^{3}(a_m)^{\alpha}c_m\vec{A}_m\right]$$

where

$$c_1\vec{A}_1=\frac{1}{k^2}\hat{F}\left[k_xk_z\vec{x}_k+k_yk_z\vec{y}_k+k_z^2\vec{z}_k\right],$$

$$c_2\vec{A}_2=\frac{1}{2k^2}\hat{F}\left[(-ikk_y-k_xk_z)\vec{x}_k+(ikk_x-k_yk_z)\vec{y}_k+\left(k^2-k_z^2\right)\vec{z}_k\right], \tag{22}$$

$$c_3\vec{A}_3=\frac{1}{2k^2}\hat{F}\left[(ikk_y-k_xk_z)\,\vec{x}_k+(-ikk_x-k_yk_z)\,\vec{y}_k+\left(k^2-k_z^2\right)\vec{z}_k\right]\ .$$

After transformations

$$L^\alpha\left(\overset{\wedge}{\vec{E}}\right)=i^\alpha\delta_{0\alpha}\frac{1}{k^2}\hat{F}\left[k_xk_z\vec{x}_k+k_yk_z\vec{y}_k+k_z^2\vec{z}_k\right]+$$

$$+(ik)^\alpha\frac{1}{k^2}\hat{F}\left[\left(\sin\left(\frac{\pi\alpha}{2}\right)kk_y-\cos\left(\frac{\pi\alpha}{2}\right)k_xk_z\right)\vec{x}_k+\right.$$

$$\left.+\left(-\sin\left(\frac{\pi\alpha}{2}\right)kk_x-\cos\left(\frac{\pi\alpha}{2}\right)k_yk_z\right)\vec{y}_k+\cos\left(\frac{\pi\alpha}{2}\right)\left(k^2-k_z^2\right)\vec{z}_k\right]\ .$$

Taking into account, that the Fourier transformation

$$\hat{F}\left(k_x,k_y,k_z\right)=F_k\left(e^{iax+iby+icz}\right)=\delta\left(k_x-a\right)\delta\left(k_y-b\right)\delta\left(k_z-c\right)$$

we have the expression for the fractional rotor

$$rot^\alpha\left[\vec{z}e^{iax+iby+icz}\right]=\frac{1}{k^2}\left(i^\alpha\delta_{0\alpha}ac+(ik)^\alpha\sin\left(\frac{\pi\alpha}{2}\right)kb-(ik)^\alpha\cos\left(\frac{\pi\alpha}{2}\right)ac\right)F\vec{x}+$$

$$+\frac{1}{k^2}\left(i^\alpha\delta_{0\alpha}bc-(ik)^\alpha\sin\left(\frac{\pi\alpha}{2}\right)ka-(ik)^\alpha\cos\left(\frac{\pi\alpha}{2}\right)bc\right)F\vec{y}+$$

$$+\frac{1}{k^2}\left(i^\alpha\delta_{0\alpha}c^2+(ik)^\alpha\cos\left(\frac{\pi\alpha}{2}\right)\left(a^2+b^2\right)\right)F\vec{z}\ . \tag{23}$$

Consider some particular cases.

If we take $c=0$ in formula (23), then

$$rot^\alpha\left[\vec{z}e^{iax+iby}\right]=$$

$$=\left(\frac{(ik)^\alpha}{k}\sin\left(\frac{\pi\alpha}{2}\right)b\vec{x}-\frac{(ik)^\alpha}{k}\sin\left(\frac{\pi\alpha}{2}\right)a\vec{y}+(ik)^\alpha\cos\left(\frac{\pi\alpha}{2}\right)\vec{z}\right)e^{iax+iby} \tag{24}$$

that coincides with expression (21) for two-variables function.

3. Some applications.

3.1. The plane wave propagation. Apply the obtained result for simplest case of propagating plane wave in medium [7]. Let plane wave propagates in the medium, characterized by a wave vector $\vec{k}$, at an angle of φ to the plane $x-z$. Electric field is given as $\vec{E}=\vec{z}e^{ik(x\cos\varphi+y\sin\varphi)}$.

Following the fractional duality principle [1], we find the new fractional field

$$\vec{E}^\alpha=\frac{1}{(ik)^\alpha}rot^\alpha\vec{E},\ \eta\vec{H}^\alpha=\frac{1}{(ik)^\alpha}rot^\alpha\left(\eta\vec{H}\right)$$

Using representation (21) we obtain

$$\vec{E}^\alpha=\frac{1}{(ik)^\alpha}rot^\alpha\vec{E}=$$

$$=\left(\sin\frac{\pi\alpha}{2}\sin\varphi\vec{x}-\sin\frac{\pi\alpha}{2}\cos\varphi\vec{y}+\cos\frac{\pi\alpha}{2}\vec{z}\right)e^{ik(x\cos\varphi+y\sin\varphi)}. \tag{25}$$

The magnetic field is found from Maxwell equation as

$$\eta\vec{H}^{\alpha} = \frac{1}{ik} rot\vec{E}^{\alpha}$$

We'll analyze the obtained expression for $\left(\vec{E}^{\alpha}, \eta\vec{H}^{\alpha}\right)$.

$\left(\vec{E}^{\alpha}, \eta\vec{H}^{\alpha}\right)$ satisfies the Maxwell equation and represents the plane propagating wave.

Introduce the local CK (x', y', z') at some point (x, y, z) by the following way:

Direct the axis z' along the vector of propagation of plane wave, i.e. $\vec{z}' = \vec{x}\cos\varphi + \vec{y}\sin\varphi$;

the axis x' coincides with the axis $z : \vec{x}' = \vec{z}$;

the axis y' is chosen such, that three vectors x', y', z' form the orthogonal system of vectors, i.e. we take $\vec{y}' = \vec{x}' \times \vec{z}' = -\vec{x}\sin\varphi + \vec{y}\cos\varphi$.

In this local CK the initial vector of voltage of the electric field $\vec{E}$ will be written as $\vec{E} = E_x\vec{x}'$ where $E_{x'} = e^{ik(x\cos\varphi + y\sin\varphi)}$.

The fractional electric field has the coordinates $\vec{E}^{\alpha} = E^{\alpha}_{x'}\vec{x}' + E^{\alpha}_{y'}\vec{y}'$, where

$$E^{\alpha}_{x'} = \cos\frac{\pi\alpha}{2} e^{ik(x\cos\varphi + y\sin\varphi)}, \quad E^{\alpha}_{y'} = -\sin\frac{\pi\alpha}{2} e^{ik(x\cos\varphi + y\sin\varphi)} .$$

So, the action of fractional rotor operator on plane wave may be interpreted as a rotation in the phase for angle $\pi\alpha/2$.

3.2. The filament of current. Consider the action of fractional rotor in radiation problems [8]. For this reason let's consider the field $\left(\vec{E}, \eta\vec{H}\right)$, radiated by filament of electric current, located along the axis z. The density of electric current is defined as $\vec{j}_e = \vec{z}J_e\delta(x)\delta(y)$.

The field will be written as

$$E_x = E_y = 0, \quad E_z(x, y) = -\frac{k\eta}{4\pi} J_e \int\limits_{-\infty}^{+\infty} \frac{Q}{\sqrt{1-\beta^2}} d\beta ,$$

$$\eta H_x = -\frac{k\eta}{4\pi} J_e \int\limits_{-\infty}^{+\infty} Q d\beta, \quad \eta H_y = \frac{k\eta}{4\pi} J_e \int\limits_{-\infty}^{+\infty} \frac{Q\beta}{\sqrt{1-\beta^2}} d\beta, \ \eta H_z = 0$$

where it is denoted $Q = Q(x, y, \beta) = e^{ik\left(\beta x + \sqrt{1-\beta^2}|y|\right)}$.

Let us apply to this field the fractional rotor and obtain the new fractional field

$$\vec{E}^{\alpha} = \frac{1}{(ik)^{\alpha}} rot^{\alpha}\vec{E}, \quad \eta\vec{H}^{\alpha} = \frac{1}{(ik)^{\alpha}} rot^{\alpha}\left(\eta\vec{H}\right)$$

or by coordinates

$$E^{\alpha}_x = -AJ_e\frac{k\eta}{4\pi} \int\limits_{-\infty}^{+\infty} Q d\beta, \quad E^{\alpha}_y =$$

$$= AJ_e \frac{k\eta}{4\pi} \int\limits_{-\infty}^{+\infty} \frac{Q}{\sqrt{1-\beta^2}} d\beta, \; E_z^\alpha = -BJ_e \frac{k\eta}{4\pi} \int\limits_{-\infty}^{+\infty} \frac{Q}{\sqrt{1-\beta^2}} d\beta$$

$$\eta H_x^\alpha = -BJ_e \frac{k\eta}{4\pi} \int\limits_{-\infty}^{+\infty} Q d\beta, \quad \eta H_y^\alpha =$$

$$= BJ_e \frac{k\eta}{4\pi} \int\limits_{-\infty}^{+\infty} \frac{Q}{\sqrt{1-\beta^2}} d\beta, \; \eta H_z^\alpha = AJ_e \frac{k\eta}{4\pi} \int\limits_{-\infty}^{+\infty} \frac{Q}{\sqrt{1-\beta^2}} d\beta$$

It is easy to see, that the obtained fractional field represents the field, radiated by combination of filaments of electric and magnetic currents with the densities

$$\vec{j}_e^\alpha = \vec{z} J_e \cos \frac{\pi\alpha}{2} \delta(x) \delta(y), \;\; \vec{j}_m^\alpha = \vec{z} J_e \sin \frac{\pi\alpha}{2} \delta(x) \delta(y).$$

It is interesting to remark, that as $\alpha = 1$ the fractional field passes to the field, radiated by filaments of magnetic current with the density $\vec{j}_m = \vec{z} J_e \delta(x) \delta(y)$.

3.3. The current sheet. In the following example, we'll consider the field $\left(\vec{E}, \eta\vec{H}\right)$, radiated by sheet of electric current, distributed in the plane $x - z$ with the density

$$\vec{j}_e = \vec{x} J_e e^{-i\psi_0} e^{i\beta_0 x} \delta(y)$$

where J_e is an amplitude, ψ_0 is an initial phase, β_0 is a coefficient of propagation.

The field of radiation has the form [8]

$$\eta\vec{H}(0,0,H_z) = \mp\frac{1}{2}\eta J_e e^{i\beta_0 x} e^{\pm i\gamma y} \vec{z}\,, \;\; \vec{E}(E_x, E_y, 0) = \frac{1}{2}\frac{\eta}{k} J_e e^{i\beta_0 x} e^{\pm i\gamma y} \left(-\gamma\vec{x} \mp \beta_0 \vec{y}\right)$$

where it is denoted $\gamma = \sqrt{k^2 - \beta_0^2}$, $\beta_0 = k\cos\phi$, $\gamma = k\sin\phi$;;

the upper sign is chosen at $y < 0$, the lower - at $y > 0$.

The fractional field $\left(\vec{E}^\alpha, \eta\vec{H}^\alpha\right)$ is defined as

$$\vec{E}^\alpha = \frac{1}{(ik)^\alpha \, rot^\alpha} \vec{E}, \;\; \eta\vec{H}^\alpha = \frac{1}{(ik)^\alpha} rot^\alpha \left(\eta\vec{H}\right)$$

Using representation (21) for fractional rotor, the fractional field will be written in the form:

$$\vec{E}^\alpha \left(E_x^\alpha, E_y^\alpha, E_z^\alpha\right) =$$

$$= \frac{1}{2}\eta J_e e^{i\beta_0 x} e^{\pm i\gamma y} \left(\cos\left(\frac{\pi\alpha}{2}\right)\sin\phi\vec{x} \mp \cos\left(\frac{\pi\alpha}{2}\right)\cos\phi\vec{y} \mp \sin\left(\frac{\pi\alpha}{2}\right)\vec{z}\right)$$

$$\eta\vec{H}^\alpha \left(H_x^\alpha, H_y^\alpha, H_z^\alpha\right) =$$

$$= \frac{1}{2}\eta J_e e^{i\beta_0 x} e^{\pm i\gamma y} \left(-\sin\left(\frac{\pi\alpha}{2}\right)\sin\phi\vec{x} \pm \sin\left(\frac{\pi\alpha}{2}\right)\cos\phi\vec{y} \mp \cos\left(\frac{\pi\alpha}{2}\right)\vec{z}\right).$$

The obtained fractional field is a field, radiated by the combination of sheets of electric and magnetic currents with the densities

$$\vec{j}_e^\alpha = \vec{x} J_e \cos\frac{\pi\alpha}{2} e^{-i\psi_0} e^{i\beta_0 x} \delta(y)\;, \quad \vec{j}_m^\alpha = \vec{x} J_e \sin\frac{\pi\alpha}{2} e^{-i\psi_0} e^{i\beta_0 x} \delta(y)\;.$$

References

[1]. Engheta N. *Fractional curl operator in electromagnetics.* Microwave Optical Techn. Lett., 17(2), pp.86-91, 1998.

[2]. Namias V. *The fractional order Fourier Transform and its application to quantum mechanics.* J. Inst. Math. Appl., v.25, 1980, pp.241-265.

[3]. Lohmann A.W. *Image rotation, Wigner rotation and the fractional Fourier transform.* J.Opt. Soc. Amer. A, v.10, Oct. 1993, pp.2181-2186.

[4]. Ozaktas H.M., Mendlovic D. *Fractional Fourier optics.* J. Opt. Soc. Am. A, v.12, No 4, April 1995,

[5]. Namias V. *Fractionalization of Hankel transforms.* Academic Press Inc. (London) Limited, 1980.

[6]. Ishimaru A. *Electromagnetic wave propagation, radiation, and scattering, Prentice-Hall.* Englewood, 1991.

[7]. Weinstein L.A. *Electromagnetic waves.* M., 1957. (Russian)

[8]. Markov G.T., Petrov B.M., Grudinskaya G.P. *Electrodynamics and propagation of radiowaves.* M., "Sovetskoye radio", 1979. (Russian)

[9]. Clemmow P.C. *The plane wave spectrum representation of electromagnetic fields.* Pergamon Press, 1966.

[10]. Samco S.G., Kilbas A.A., Marichev O.I. *Integrals and fractional order derivatives and some their applications.* Minsk, "Nauka i tekhnika", 1987. (Russian)

[11]. Oldham K.B., Spanier J. *The fractional calculus.* N.Y., London, Acad. Press, 1974

[12]. Kolmogorov A.N., Fomin S.V. *The elements of theory of functions and functional analyses.* M., "Nauka", 1976. (Russian)

Turab M. Akhmedov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 492 82 20 (off.)

Received October 21, 2004; Revised December 28, 2004.
Translated by Mamedova Sh.N.