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COMPLETENESS AND MINIMALITY OF ONE SYSTEM OF EXPONENTS IN SPACE OF PIECEWISE CONTINUOUS FUNCTIONS

Abstract

In the paper the space conjugate to the space of the piecewise continuous functions $KC[-\pi,\pi]$ is constructed and the necessary and sufficient conditions were found for completeness and minimality of the following system of exponents:

$$\left\{e^{i[(n+\alpha_1)t+\beta(t)]};\ e^{-i[(k+\alpha_2)t+\beta(t)]}\right\}_{n=0;k=1}^{\infty}$$

where

$$\beta(t)=\begin{cases}\beta_1, & -\pi\le t<0,\\ \beta_2, & 0\le t\le\pi,\end{cases}$$

$\alpha_i,\beta_i\in R,\ i=\overline{1,2}$ *are real parameters.*

Consider the following system of exponents:

$$\left\{e^{i[(n+\alpha_1)t+\beta(t)]};\ e^{-i[(k+\alpha_2)t+\beta(t)]}\right\}_{n=0;k=1}^{\infty}, \tag{1}$$

where

$$\beta(t)=\begin{cases}\beta_1, & \pi\le t<0,\\ \beta_2, & 0\le t\le\pi,\end{cases}$$

$\alpha_i,\beta_i\in R,\ i=\overline{1,2}$ are real parameters.

In 1950 A.B.Bitsadze [1] proved the uniform convergence of biorthogonal series by the system $\{\sin nx+\cos nx\}_{n=1}^{\infty}$ for functions from Holder class $C^{\alpha}[0,\pi]$.Then S.M.Ponamarev applied [2] this method for the system $\left\{\sin\left(n-\frac{1}{4}\right)x\right\}_{n=1}^{\infty}$. The completeness in $C[a,b]$ of the systems $\{\mathrm{Re}[\varphi^n(t)]\}_{n=0}^{\infty}$, $\{[\mathrm{Im}[\varphi^n(t)]]\}_{n=0}^{\infty}$ and $\{\mathrm{Re}[\varphi^n(t)];\mathrm{Im}[\varphi^n(t)]\}_{n=0}^{\infty}$ has considered Yu.A. Kazmin [3-5]. First the necessary and sufficient condition of completeness and minimality of sine systems in $C[0,\pi]$ was found by E.I.Moiseev [6,7]. A.I.Sedletsckii [8] considered the completeness of the system of exponents, cosine and sine in corresponding spaces of continuous functions.

In B.T.Bilalov paper [9] the necessary and sufficient condition of completeness and minimality of the system of functions $\left\{A(t)e^{int};B(t)e^{-ikt}\right\}_{n=0;k=1}^{\infty}$, $\{A(t)\varphi^n(t);B(t)\bar{\varphi}^n(t)\}_{n=0}^{\infty}$ was found in spaces $C[-\pi,\pi]$ and $C[a,b]$, respectively, where $A(t)$, $B(t)$, $\varphi(t)$ are complex-valued functions.

In the present paper the completeness and minimality of system (1) in the space of piecewise continuous functions is considered.

Introduce the following notation:

$$\begin{aligned}\mu_1 &= -(\alpha_1+\alpha_2)+(\beta_1-\beta_2)/\pi;\\ \mu_2 &= (\beta_2-\beta_1)/\pi;\end{aligned}$$

$C[-\pi,-0)$ is a space of bounded continuous functions determined in half –interval $[-\pi,0)$ with the norm

$$\|f\|=\sup_{-\pi\le t<0}|f(t)|. \tag{2}$$

$KC[-\pi,\pi]$ is a space of piecewise continuous functions that have at the point $t=0$ the break of the first kind $(f(0)=f(+0))$ with the norm

$$\|f\|=\left[\left(\sup_{-\pi\le t<0}|f(t)|\right)^2+\left(\max_{0\le t\le\pi}|f(t)|\right)^2\right]^{1/2} \tag{3}$$

$KC_A[-\pi,\pi]=\left\{f:\ f\in KC[-\pi,\pi],\ f(-\pi)A^{-1}(-\pi)=f(\pi)A^{-1}(\pi)\right\}$, here $A(t)$ is some function from the class $KC[-\pi,\pi]$.

N is a set of natural numbers.

$B_1\times B_2$ is the Cartesian product of Banach space in which the norm is determined by the following form

$$\|(f,g)\|=\left(\|f\|^2+\|g\|^2\right)^{1/2},\ f\in B_1,\ g\in B_2 \tag{4}$$

$V^0[a,b]$ is a space of functions of bounded variations which vanishes at the point a.

It is evident that the space $KC[-\pi,-0]$ and $C[-\pi,-0)\times C[0,\pi]$ is isometrically isomorphic. Therefore they can be identified:

$$KC[-\pi,\pi]=C[-\pi,-0)\times C[0,\pi]$$

The spaces $C[-\pi,-0)$ and $C[-\pi,0)$ are also isometrically isomorphic, therefore we can also identify them:

$$C[-\pi,-0)=C[-\pi,0].$$

In the paper of T.Kato [10, p.209] it was shown that Cartesian product of the Banach spaces $B_1\times B_2$ is a Banach space by the norm (4), moreover $(B_1\times B_2)^*=B_1^*\times B_2^*$.

Taking into account this fact we can assert that

$$(KC[-\pi,\pi])^*=(C[-\pi,0])^*\times(C[0,\pi])^*.$$

It is known that the space conjugate to the space $C[a,b]$ of continuous functions is $V^0[a,b]$. Therefore

$$(KC[-\pi,\pi])^*=V^0[-\pi,0]\times V^0[0,\pi].$$

Let's formulate now the basic result of this paper.

Theorem. *a) Let $0\le\mu_2<1$, then system (1) is minimal in $KC[-\pi,\pi]$ iff $\mu_1<1$;*

b) let $0<\mu_2<1,\ \mu_1>0$ and $\mu_1\overline{\in}\mathbf{N}$ then system (1) is complete in $KC[-\pi,\pi]$;

c) let $0<\mu_2<1$ then system (1) is compact in $KC_A[-\pi,\pi]$ iff $\mu_1\ge 0$;

d) let $0\le\mu_2<1,\ \mu_1<0$ then system (1) is not complete in $KC[-\pi,\pi]$.

Proof. At first let's prove that at $0\le\mu_i<1,\ i=1,2$ system (1) is minimal in $KC[-\pi,\pi]$. We'll prove this assertion by virtue of previously obtained obvious form of the biorthogonal system $\left\{h_k^+(\theta);\ h_k^-(\theta)\right\}_{n=0;k=1}^{\infty}$ to system (1):

$$h_n^+(\theta)=\frac{1}{2\pi}\left(e^{i\theta}+1\right)_{-1}^{-\mu_1}\left(e^{i\theta}-1\right)_{+1}^{-\mu_2}e^{-i(\alpha_1\theta+\beta(\theta)+\beta_1-\beta_2)}\times$$

$$\times\sum_{m=0}^{n}(-1)^{n-m}C_{\mu_2}^{n-m}\sum_{s=0}^{m}C_{\mu_1}^{m-s}e^{-is\theta},\quad n\ge 0$$

$$h_k^-(\theta)=\frac{1}{2\pi}\left(e^{i\theta}+1\right)_{-1}^{-\mu_1}\left(e^{i\theta}-1\right)_{+1}^{-\mu_2}e^{-i(\alpha_1\theta+\beta(\theta)-2\beta_2)}\times$$

$$\times\sum_{m=1}^{k}(-1)^{k-m}C_{\mu_2}^{k-m}\sum_{s=0}^{m}C_{\mu_1}^{m-s}e^{is\theta},\quad k\ge 1.$$

where $(z\pm 1)_{\mp 1}^{\mu}$ are defined branches of the multiple-valued functions $(z\pm 1)^{\mu}$, and $C_\mu^n=\frac{\mu(\mu-1)\cdot\ldots\cdot(\mu-n+1)}{n!}$ are binomial coefficients.

Show that at $0\le\mu_i<1,\ i=1,2$ the functions $h_n^+(\theta),\ h_k^-(\theta)\quad(n\ge 0,\ k\ge 1)$ are summable.

Really,

$$\int_{-\pi}^{\pi}\left|h_n^+(\theta)\right|d\theta\le C_n\cdot\int_{-\pi}^{\pi}\left|e^{i\theta}-1\right|^{-\mu_1}\cdot\left|e^{i\theta}+1\right|^{-\mu_2}d\theta=$$

$$=C_n'\int_{-\pi}^{\pi}\left|\cos\frac{\theta}{2}\right|^{-\mu_1}\cdot\left|\sin\frac{\theta}{2}\right|^{-\mu_2}d\theta=C_n'\left(\int_{-\pi}^{0}\left|\cos\frac{\theta}{2}\right|^{-\mu_1}\cdot\left|\sin\frac{\theta}{2}\right|^{\mu_2}d\theta+\right.$$

$$\left.+\int_{0}^{\pi}\left|\cos\frac{\theta}{2}\right|^{-\mu_1}\cdot\left|\sin\frac{\theta}{2}\right|^{-\mu_2}d\theta\right).$$

where $n\ge 0,\ C_n$ and C_n' are the constants depending only on n.

It is obvious that

$\left|\sin\frac{\theta}{2}\right|>\left|\frac{\theta}{4}\right|$ near the point zero,

$\left|\cos\frac{\theta}{2}\right|>\left|\frac{\pi}{4}+\frac{\theta}{4}\right|$ near the point $-\pi$,

$\left|\cos\frac{\theta}{2}\right|>\left|\frac{\pi}{4}-\frac{\theta}{4}\right|$ near the point π.

Then

$$\int_{-\pi}^{0}\left|\cos\frac{\theta}{2}\right|^{-\mu_1}\left|\sin\frac{\theta}{2}\right|^{-\mu_2}d\theta=\int_{-\pi}^{-\pi/2}\left|\cos\frac{\theta}{2}\right|^{-\mu_1}\left|\sin\frac{\theta}{2}\right|^{-\mu_2}d\theta+$$

$$+\int_{-\pi/2}^{0}\left|\cos\frac{\theta}{2}\right|^{-\mu_1}\left|\sin\frac{\theta}{2}\right|^{-\mu_2}d\theta\le C\left(\int_{-\pi}^{-\pi/2}\left|\frac{\theta}{4}+\frac{\pi}{4}\right|^{-\mu_1}d\theta+\int_{-\pi/2}^{0}|\theta|^{-\mu_2}d\theta\right).$$

Since $0\le\mu_i<1,\ i=1,2$ then the integrals on the right hind side of the inequality sign converge.

Hence, we obtain the convergence of the integral $\int_{-\pi}^{0}\left|\cos\frac{\theta}{2}\right|^{-\mu_1}\left|\sin\frac{\theta}{2}\right|^{-\mu_2}d\theta$.

Similarly, we can show the convergence of the integral $\int_{0}^{\pi}\left|\cos\frac{\theta}{2}\right|^{-\mu_1}\left|\sin\frac{\theta}{2}\right|^{-\mu_2}d\theta$.

Hence it follows the convergence of the integral $\int_{-\pi}^{\pi}\left|h_n^+(\theta)\right|d\theta,\ \forall n\ge 0$.

Completely similarly the convergence of the integrals $\int\limits_{-\pi}^{\pi} \left|h_k^-(\theta)\right| d\theta,\ \forall k \geq 1$ is proved.

Thus we obtain the summation of the system of the functions $\left\{h_n^+(\theta);\ h_k^-(\theta)\right\}_{n=0;k=1}^{\infty}$.

Then it is evident that the functions

$$\omega_n^+(\theta) = \begin{cases} \int\limits_{-\pi}^{\theta} h_n^+(t)\,dt, & -\pi \leq \theta < 0, \\ \int\limits_{0}^{\theta} h_n^+(t)\,dt, & 0 \leq \theta \leq \pi, \qquad n \geq 0 \end{cases}$$

$$\omega_k^-(\theta) = \begin{cases} \int\limits_{-\pi}^{\theta} h_k^-(t)\,dt, & -\pi \leq \theta < 0, \\ \int\limits_{0}^{\theta} h_k^-(t)\,dt, & 0 \leq \theta \leq \pi, \qquad k \geq 1. \end{cases}$$

are absolutely continuous and belong to the space $(KC[-\pi,\pi])^*$ and moreover

$$\int\limits_{-\pi}^{\pi} e^{i[(n+\alpha_1)t+\beta(t)]} d\omega_k^+(t) = \delta_{nk},\ n \geq 0,\ k \geq 0$$

$$\int\limits_{-\pi}^{\pi} e^{i[(n+\alpha_1)t+\beta(t)]} d\omega_k^-(t) = 0, \quad n \geq 0,\ k \geq 1$$

$$\int\limits_{-\pi}^{\pi} e^{-i[(n+\alpha_2)t+\beta(t)]} d\omega_k^-(t) = \delta_{nk}, \quad n \geq 1,\ k \geq 1$$

$$\int\limits_{-\pi}^{\pi} e^{-i[(n+\alpha_2)t+\beta(t)]} d\omega_k^+(t) = 0, \quad n \geq 1,\ k \geq 0.$$

As a result we obtain that at $0 \leq \mu_i < 1$ system (1) is minimal in $i = 1,2$. And now let $-1 \leq \mu_1 < 0$. In this case from the previous discussions it follows that the system

$$\left\{e^{i[n+\alpha_1-1]t+\beta(t)};\ e^{-i[(k+\alpha_2)t+\beta(t)]}\right\}_{n=0;\ k=1}^{\infty} \tag{1'}$$

is minimal. Really, if we denote $\alpha_1' = \alpha_1 - 1$ then for system $(1')$

$$\mu_1' = -\left(\alpha_1' + \alpha_2\right) + \frac{\beta_1 - \beta_2}{\pi} = \mu_1 + 1$$

$$\mu_2' = \frac{\beta_2 - \beta_1}{\pi} = \mu_2$$

besides $0 \leq \mu_i' < 1,\ i = 1,2$. From the previous discussions it follows that system $(1')$ is also minimal in $KC[-\pi,\pi]$ and it means that system (1) is minimal in $KC[-\pi,\pi]$. Continuing this process we obtain the minimality of system (1) in $KC[-\pi,\pi]$ at $\mu_1 < 1,\ 0 \leq \mu_2 < 1$.

Consider the completeness of system (1). Take any piecewise Hölder function $f(t)$ determined on the segment $[-\pi,\pi]$ and having break at the point zero, moreover $f(0)=f(+0)$. We know that under the condition $0<\mu_i<1,\ i=1,2$ the biorthogonal series by system (1) uniformly converges to $f(t)$ on the segment $[-\pi,\pi]$.

Since the set of piecewise Hölder functions is compact in the space of piecewise continuous functions by the norm $KC[-\pi,\pi]$, we obtain that system (1) is complete in $KC[-\pi,\pi]$. Let's now $1<\mu_1<2$.

Consider the completeness of the system

$$\left\{e^{i[(n+\alpha_1+1)t+\beta(t)]};\ e^{-i[(k+\alpha_2)t+\beta(t)]}\right\}_{n=0;k=1}^{\infty} \qquad (1'')$$

If we denote $\alpha_1''=\alpha_1+1$ then for system $(1'')$

$$\mu_1'' = -\left(\alpha_1''+\alpha_1\right)+\frac{\beta_1-\beta_2}{\pi}=\mu_1-1$$

$$\mu_2'' = \frac{\beta_2-\beta_1}{\pi}=\mu_2$$

besides $0<\mu_i''<1\ (i=1,2)$. From previous discussion it follows that system $(1'')$ is also complete in $KC[-\pi,\pi]$ and it means that system (1) is also complete in $KC[-\pi,\pi]$. Continuing this process we obtain the completeness of system (1) under the condition $\mu_1>0,\ \mu_1\overline{\in}\mathbf{N}$ and $0<\mu_2<1$.

So we obtain that at $1<\mu_1<2$ and $0<\mu_2<1$ system $(1'')$ is complete. Since $e^{i[\alpha_1 t+\beta(t)]}\in KC[-\pi,\pi]$ then hence we obtain that system (1) at $0<\mu_2<1$ and $1<\mu_2<2$ is not minimal in $KC[-\pi,\pi]$. So, continuing this process we obtain non-minimality of system (1) at $\mu_1>0,\ \mu_1\overline{\in}\mathbf{N}$ and $0<\mu_2<1$.

If $0<\mu_2<1$ and $\mu_1=0$ then we know that for any piecewise Hölder function $f(t)$ from the class $KC_A[-\pi,\pi]$ where $A(t)=e^{i[\alpha_1 t+\beta(t)]},\ f(0)=f(+0)$ the biorthogonal series by system (1) uniformly converges to $f(t)$ on the segment $[-\pi,\pi]$. Since the set of piecewise Hölder functions from the class $KC_A[-\pi,\pi]$ is compact in $KC_A[-\pi,\pi]$. We obtain that system (1) is compact in $KC_A[-\pi,\pi]$.

Let now $\mu_1=1,\ 0<\mu_2<1$. Then system $(1'')$ is compact in $KC_{A''}[-\pi,\pi]=KC_A[-\pi,\pi]$ where $A''(t)=e^{i(\alpha_1 t+\beta(t)+t)}$.

It means that system (1) is compact in $KC_A[-\pi,\pi]$. Continuing this process we obtain that at $\mu_1\in\mathbf{N}$ and $0<\mu_2<1$ system (1) is compact in $KC_A[-\pi,\pi]$.

So, we show that at $\mu_1=1,\ 0<\mu_2<1$ system (1) is compact in $KC_A[-\pi,\pi]$ then hence we obtain that system (1) is not minimal in $KC[-\pi,\pi]$, because $A(t)\in KC_A[-\pi,\pi]$. As a result continuing this process we obtain non-minimality of system (1) in $KC[-\pi,\pi]$ at $\mu_1\in\mathbf{N},\ 0<\mu_2<1$.

At $\mu_2=0$ in B.T.Bilalov's paper [9, p.73] it was proved that system (1) is non-minimal in $C[-\pi,\pi]$ at $\mu_1\geq 1$, it means that it is non-minimal in $KC[-\pi,\pi]$.

Consider the case when $-1\leq\mu_1<0,\ o\leq\mu_2<1$. As we've shown in this case system $(1')$ is minimal in $KC[-\pi,\pi]$. Hence we obtain that the function $e^{i[(\alpha_1-1)t+\beta(t)]}\in KC_A[-\pi,\pi]$ shouldn't be approximated by the linear combinations of system (1). Hence it follows that system (1) is not compact in $KC_A[-\pi,\pi]$. Continuing this process we obtain that at $0<\mu_2<1,\ \mu_1<0$ system (1) isn't compact in $KC_A[-\pi,\pi]$ and so it is not complete in $KC[-\pi,\pi]$.

At $\mu_2=0$ non-compactness of system (1) in $KC_A[-\pi,\pi]$ is evident. Because in this case all members of system (1) will be continuous functions. It is clear that any discontinuous functions shouldn't be approximated by continuous functions by the norm $KC[-\pi,\pi]$. From the non-compactness of system (1) in $KC_A[-\pi,\pi]$ we obtain non-completeness in $KC[-\pi,\pi]$.

The theorem is proved.

References

[1]. Bitsadze A.V. *On one system of functions.* Uspekhi Math. Nauk, 1950, v.5, No4(38), pp.150-151. (Russian)

[2]. Ponamarev S.M. *On one eigen-values problem.* Soviet Doklady, 1979, v.249, No5, pp.1068-1070. (Russian)

[3]. Kazmin Yu.A. *On closure of linear span of two systems of functions.* Soviet Doklady, 1977, v.236, No3, pp.535-537. (Russian)

[4]. Kazmin Yu.A. *Closure of linear span of one system of functions.* Sibirskii mathem. zhurnal. 1977, v.18, No4, pp.799-805. (Russian)

[5]. Kazmin Yu.A. *On nontrivial solutions of homogeneous Abel problem.* Math. sb., 1979, 109(115), No2(6), pp.254-274. (Russian)

[6]. Moiseev E.I. *On basicity of sine and cosine systems.* Soviet Dokl, 1984, v.275, No4, pp.794-798. (Russian)

[7]. Moiseev E.I. *On basicity of one system of sines.* Differential Uravneniya. 1987, v.23, No1, pp.794-798. (Russian)

[8]. Sedletskii A.M. *On convergence of non-harmonic Fourier series by the systems of exponent, cosine and sine.* Soviet Doklady, 1988, v.301, No5, pp.1053-1056. exponent

[9]. Bilalov B.T. *Basic properties of system of eigen-functions of some differential operators and their generalizations.* Dis. of doct. MSU, 1995, 210p. (Russian)

[10]. Kato T. T*heory of perturbations of linear operators.* "Mir", 1972, 740p. (Russian)

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