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ON CORRECT SOLVABILITY OF A SECOND ORDER BOUNDARY-VALUE PROBLEM FOR ONE CLASS OF OPERATOR-DIFFERENTIAL EQUATION ON FINITE SEGMENT

Abstract

In the work algebraic conditions have been found on the coefficients of one class of second-order operator-differential equation, which provide single valued and correct solvability of second boundary-value problem on finite segment.

In the separable Hilbert space H we'll consider the operator-differential equation:

$$-\left(d/dt-\omega_1 A\right)\left(d/dt-\omega_2 A\right)u\left(t\right)+\sum_{j=0}^{1}A_{2-j}u^{(j)}\left(t\right)=f\left(t\right),t\in\left(0,1\right) \qquad (1)$$

and initial-boundary condition

$$u'\left(0\right)=0,\ \ u'\left(1\right)=0, \qquad (2)$$

where $f\left(t\right),\ u\left(t\right)$ are vector-valued functions, derivatives are given in the sense of theory of distributions [1], ω_1,ω_2 are real numbers $(\omega_1<0,\omega_2>0)$, A is a positive definite selfadjoint operator, $A_1,\ A_2$ are linear operators in H.

Let us denote by H_γ a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma=D\left(A^\gamma\right),\left(x,y\right)_\gamma=\left(A^\gamma_x,A^\gamma_y\right)\ \ (\gamma\geq 0)$.

Denote by $L_2\left((0,1);H\right)$ a Hilbert space of vector-functions, defined in the interval $(0,1)$ with values from H_γ, measurable and quadratically integrable by Bochner, moreover

$$\|f\|_{L_2((0,1);H_\gamma)}=\left(\int_0^1\|f\left(t\right)\|^2_\gamma\,dt\right)^{1/2}.$$

Then we'll define the Hilbert spaces [1]

$$W_2^2\left((0,1);H\right)=\{u:u''\in L_2\left((0,1);H\right),A^2u\in L_2\left((0,1);H\right)\},$$

$$\mathring{W}_2^2\left((0,1);H\right)=\{u:u\in W_2^2\left((0,1);H\right),u'\left(0\right)=u'\left(1\right)=0\}.$$

Definition 1. *If the vector-function* $u\left(t\right)\in W_2^2\left((0,1);H\right)$ *satisfies equation (1) in the interval* $(0,1)$ *almost everywhere, then we'll call it a regular solution of equation (1).*

Definition 2. *If at any* $f\left(t\right)\in L_2\left((0,1);H\right)$ *there exists a regular solution of equation (1), which satisfies boundary condition (2) in the sense*

$$\lim_{t\to+0}\left\|u'\left(t\right)\right\|_{1/2}=0,\ \lim_{t\to-1}\left\|u'\left(t\right)\right\|_{1/2}=0,$$

and the inequality

$$\|u\|_{W_2^2((0,1);H)}\leq const\,\|f\|_{L_2((0,1);H)},$$

holds, then we'll call problem (1)-(2) regular solvable.

In the given paper we'll indicate the sufficient conditions on coefficients of operator-differential equation (1), which provides a regular solvability of problem (1), (2).

Note, that a regular solvability of equation (1) with boundary condition $u(0) = u(1) = 0$ has been investigated by the author in the work [2].

Define in the space $\mathring{W}_2^2((0,1);H)$ the following operators

$$P_0 u = -(d/dt - \omega_1 A)(d/dt - \omega_2 A) u, \ u \in \mathring{W}_2^2((0,1);H),$$

$$P_1 u = A_1 \frac{du}{dt} + A_2 u, \quad u \in \mathring{W}_2^2((0,1);H),$$

and

$$Pu = P_0 u + P_1 u, \ u \in \mathring{W}_2^2((0,1);H)$$

it holds.

Lemma 1. *Let A be a positive definite selfadjoint operator, $\omega_1 < 0$, $\omega_2 > 0$. Then the operator P_0 isomorphically maps the space $\mathring{W}_2^2((0,1);H)$ onto $L_2((0,1);H)$.*

Proof. It is obvious, that the equation $P_0 u = 0$ has the general solution

$$u_0(t) = e^{\omega_1 t A} g_0 + e^{\omega_2 (t-1) A} g_1$$

from the space $W_2^2((0,1);H)$, where $g_0, g_1 \in H_{3/2}$, and $e^{\omega_1 t A}$ and $e^{-\omega_2 t A}$ are strongly continuous subgroups, generated by the operators $\omega_1 A$ and $\omega_2 A$, respectively. From the condition $u \in \mathring{W}_2^2((0,1);H)$ and $(u'(0) = u'(1) = 0)$ we obtain the following system of equations for definition of g_0 and g_1 :

$$\begin{cases} \omega_1 A g_0 + \omega_2 A e^{-\omega_2 A} g_1 = 0, \\ \omega_1 A e^{\omega_1 A} g_0 + \omega_2 A g_1 = 0. \end{cases}$$

Hence, $g_0 = g_1 = 0$, consequently $u_0(t) = 0$. Now we'll show, that the image of the operator P_0 coincides with the space $L_2((0,1);H)$.

Consider the equation $P_0 u = f$, $\forall f \in L_2((0,1);H)$ It is easy to see, that the vector-function

$$u_1(t) = \frac{1}{2\pi} \int_{2\pi}^{+\infty} P_0^{-1}(-i\xi) \left(\int_0^1 f(s) e^{-is\xi} ds \right) e^{i\xi i} d\xi, \ t \in R = (-\infty, \infty)$$

satisfies equation (1) almost everywhere (at $t \notin (0,1)$ we take into account, that $f(t) = 0$). Show, that $u_1(t) \in W_2^2(R,H)$. Really, by Plansharel's theorem

$$\|u_1\|^2_{W_2^2(R,H)} = \|u_1''\|^2_{L_2(R+H)} + \|A^2 u_1\|^2_{L_2((0,1);H)} =$$

$$= \|\xi^2 \hat{u}_1(\xi)\|^2_{L_2} + \|A^2 \hat{u}_1(\xi)\|^2_{L_2} \le \left\| \xi^2 P_0^{-1}(-i\xi) \hat{f}(\xi) \right\|^2_{L_2} +$$

$$+ \left\| A^2 P_0^{-1}(-i\xi) \hat{f}(\xi) \right\|^2_{L_2} \le \left(\sup_{\xi \in R} \|\xi^2 P_0^{-1}(-i\xi)\|^2 + \right.$$

$$+\sup_{\xi\in R}\left\|\xi^2P_0^{-1}(-i\xi)\right\|^2\Big)\left\|\hat f(\xi)\right\|_{L_2}^2.$$

Here $\hat u_1(\xi)$, $\hat f(\xi)$ are Fourier transformations of the vector-functions $u_1(t)$ and $f(t)$, respectively. Since at any $\xi\in R$

$$\left\|\xi^2P_0^{-2}(-i\xi)\right\|=\sup_{\mu\in\sigma(A)}\left|\xi^2(i\xi+\omega_1\mu)^{-1}(i\xi+\omega_2\mu)^{-1}\right|\le$$

$$\le\sup_{\mu\in\sigma(A)}\left|\xi^2\left(\xi+\omega_1^2\mu^2\right)^{-1/2}\left(\xi^2+\omega_2^2\mu^2\right)^{-1/2}\right|\le 1$$

and

$$\left\|A^2P_0^{-1}(-i\xi)\right\|=\sup_{\mu\in\sigma(A)}\left|\mu^2(i\xi+\omega_1\mu)^{-1}(i\xi+\omega_2\mu)^{-1}\right|=$$

$$=\sup_{\mu\in\sigma(A)}\left|\mu^2\left(\xi^2+\omega_1^2\mu^2\right)^{-1/2}\left(\xi^2+\omega_2^2\mu^2\right)^{-1/2}\right|\le\frac{1}{|\omega_1\omega_2|},$$

then

$$\|u_1\|_{W_2^2(R;H)}^2\le\left(1+\frac{1}{|\omega_1\omega_2|^2}\right)\left\|\hat f(\xi)\right\|_{L_2}^2=$$

$$=\left(1+\frac{1}{|\omega_1\omega_2|^2}\right)\left\|\hat f(\xi)\right\|_{L_2}^2.$$

So, $u_1\in W_2^2(R;H)$. Let's denote by $\psi(t)$ the contraction of the vector-function $u_1(t)$ on the interval $[0,1]$. It is obvious, that $\psi(t)\in W_2^2((0,1);H)$.

Then by the traces theorem $\psi'(0)\in H_{1/2}$, $\psi'(1)\in H_{1/2}$ (see [1]).

Now we'll search the solution of the equation $P_0u=f$ in the form

$$u(t)=\psi(t)+e^{\omega_1tA}g_0+e^{\omega_2(t-1)A}g_1,$$

where $g_0,g_1\in H_{3/2}$ are desired vector functions. For their definition from boundary conditions (2) we obtain

$$\begin{cases}\omega_1Ag_0+\omega_2Ae^{-\omega_2A}g_1=-\psi'(0),\\ \omega_1e^{\omega_2A}g_0+\omega_2Ag_1=-\psi'(1).\end{cases}$$

Hence

$$g_0=\frac{1}{\omega_1}\left(E-e^{(\omega_1-\omega_2)A}\right)^{-1}\left(e^{-\omega_2A}A^{-1}\psi'(1)-A^{-1}\psi'(0)\right)\in H_{3/2},$$

$$g_1=\frac{1}{\omega_2}\left(E-e^{(\omega_1-\omega_2)A}\right)^{-1}\left(e^{\omega_1A}A^{-1}\psi'(0)-A^{-1}\psi'(1)\right)\in H_{3/2}.$$

So, $u(t)\in W_2^2((0,1);H)$. On the other hand from the theorem on intermediate derivatives it implies, that

$$\|P_0u\|_{L_2((0,1);H)}\le\left\|u''\right\|_{L_2((0,1);H)}+|\omega_1+\omega_2|\left\|Au'\right\|_{L_2((0,1);H)}+$$

$$+|\omega_1\omega_2|\left\|A^2u\right\|_{L_2((0,1);H)}\le const\,\|u\|_{W_2^2((0,1);H)}.$$

Then from Banach theorem on the inverse operator it follows the statement of the lemma.

Lemma 2. *At all* $u(t) \in \overset{\circ}{W}{}_2^2((0,1);H)$ *the following inequalities hold*

$$\left\|Au'\right\|_{L_2((0,1);H)} \leq c_1 \left\|P_0 u\right\|_{L_2((0,1);H)}, \tag{3}$$

$$\left\|u''\right\|_{L_2((0,1);H)} \leq c_2 \left\|P_0 u\right\|_{L_2((0,1);H)}, \tag{4}$$

$$\left\|A^2 u\right\|_{L_2((0,1);H)} \leq c_0 \left\|P_0 u\right\|_{L_2((0,1);H)}, \tag{5}$$

where

$$c_1 = 2^{-1}\left|\omega_1\omega_2\right|^{-1/2}, \quad c_2 = 1,$$

$$c_0 = \begin{cases} \left|\omega_1\omega_2\right|^{-1}, \text{ at } \omega_1 = -\omega_2, \\ \left|\omega_1\omega_2\right|^{-1}\left(2 + 2^{-1}\left|\omega_1+\omega_2\right|\left|\omega_1\omega_2\right|^{-1/2}\right), \text{ at } \omega_1 \neq -\omega_2. \end{cases}$$

Proof. We'll multiply the expression $P_0 u$ scalarly by $-u''$ in the space $L_2((0,1);H)$ and find the real part of the obtained expression:

$$\operatorname{Re}\left(P_0 u, -u''\right)_{L_2} = \operatorname{Re}\left(-u'' + (\omega_1+\omega_2) Au + \left|\omega_1\omega_2\right| A^2 u, -u''\right)_{L_2} =$$

$$= \left\|u''\right\|_{L_2}^2 - (\omega_1+\omega_2)\operatorname{Re}\left(Au', u''\right)_{L_2} - \left|\omega_1\omega_2\right|\operatorname{Re}\left(A^2 u, u''\right)_{L_2}. \tag{6}$$

Then, taking into account, that $u \in \overset{\circ}{W}{}_2^2((0,1);H)$, $(u'(0) = u'(1) = 0)$ after integrating by parts we obtain

$$\left(Au', u''\right)_{L_2} = -\left(u'', Au'\right)_{L_2}$$

i.e.

$$\operatorname{Re}\left(Au', u''\right)_{L_2} = 0. \tag{7}$$

Similarly we have

$$-\operatorname{Re}\left(A^2 u, u''\right)_{L_2} = \left\|Au'\right\|_{L_2}^2. \tag{8}$$

Taking into account equalities (7) and (8) in equality (6) we obtain:

$$\operatorname{Re}\left(P_0 u, -u''\right)_{L_2} = \left\|u''\right\|_{L_2}^2 + \left|\omega_1\omega_2\right|\left\|Au'\right\|_{L_2}^2. \tag{9}$$

From equality (9) it follows, that

$$\left\|u''\right\|_{L_2}^2 \leq \operatorname{Re}\left(P_0 u, -u''\right)_{L_2} \leq \left\|P_0 u\right\|_{L_2}\left\|u''\right\|_{L_2}$$

or

$$\left\|u''\right\|_{L_2} \leq \left\|P_0 u\right\|_{L_2}. \tag{10}$$

So, inequality (4) is proved. From inequality (9) we have

$$\left\|u''\right\|_{L_2}^2 + \left|\omega_1\omega_2\right|\left\|Au'\right\|_{L_2}^2 \leq \operatorname{Re}\left(P_0 u, -u''\right)_{L_2} \leq \left\|P_0 u\right\|_{L_2}\left\|u''\right\|_{L_2} \leq$$

$$\leq \frac{1}{4}\left\|P_0 u\right\|_{L_2}^2 + \left\|u''\right\|_{L_2}^2,$$

or

$$|\omega_1\omega_2| \left\|Au'\right\|^2_{L_2} \leq \frac{1}{4} \left\|P_0u\right\|^2_{L_2}.$$

Hence, we find, that

$$\left\|Au'\right\|_{L_2} \leq \frac{1}{2\,|\omega_1\omega_2|} \left\|P_0u\right\|_{L_2},$$

i.e. inequality (3) is true. Now we'll prove inequality (5).

If $\omega_1 = -\omega_2$, then multiplying $P_0\, u$ by the function $A^2u \in L_2\left((0,1)\,;H\right)$ scalarly in $L_2\left((0,1)\,;H\right)$ and integrating by parts we obtain

$$\left\|Au'\right\|^2_{L_2} + |\omega_1\omega_2| \left\|A^2u\right\|^2_{L_2} = \mathrm{Re}\left(P_0u, Au\right)_{L_2} \leq \left\|P_0u\right\|_{L_2} \left\|A^2u\right\|_{L_2},$$

i.e.

$$\left\|A^2u\right\|_{L_2} \leq |\omega_1\omega_2|^{-1} \left\|P_0u\right\|_{L_2}.$$

If $\omega_1 \neq -\omega_2$, then from the equality

$$P_0u = -u'' + (\omega_1 + \omega_2)\, Au' + |\omega_1\omega_2|\, A^2u$$

it follows, that

$$|\omega_1\omega_2| \left\|A^2u\right\|_{L_2} \leq \left\|u''\right\|_{L_2} + |\omega_1 + \omega_2| \left\|Au'\right\|_{L_2} + \left\|P_0u\right\|_{L_2} \leq$$

$$\leq \left(2 + 2^{-1} |\omega_1 + \omega_2| \, |\omega_1\omega_2|^{-1/2}\right) \left\|P_0u\right\|_{L_2},$$

i.e.

$$\left\|A^2u\right\|_{L_2((0,1);H)} \leq |\omega_1\omega_2|^{-1} \left(2 + 2^{-1} |\omega_1 + \omega_2| \, |\omega_1\omega_2|^{-1/2}\right) \left\|P_0u\right\|_{L_2}.$$

The lemma is proved.

Theorem. *Let A be a positive definite selfadjoint operator, $\omega < 0$, $\omega_2 > 0$, then operators $B_1 = A_1A^{-1}$, $B_2 = A_2A^{-2}$ be bounded in H, moreover*

$$\alpha = c_1 \left\|B_1\right\| + c_0 \left\|B_2\right\| < 1,$$

where the numbers c_1 and c_0 are defined from lemma 2. The problem (1), (2) is regular solvable.

Proof. Let's write problem (1), (2) in the form of the equation $(P_0 + P_1)\, u = f$, $f \in L_2\left((0,1)\,;H\right)$, $u \in \mathring{W}^2_2\left((0,1)\,;H\right)$ and we'll make the substitution $P_0u = v$. Then we obtain the equation $v + P_1P_0^{-1}v = f$ in the space $L_2\left((0,1)\,;H\right)$. On the other hand for any $v \in L_2\left((0,1)\,;H\right)$

$$\left\|P_1P_0^{-1}v\right\|_{L_2} = \left\|P_1u\right\|_{L_2} \leq \left\|B_1\right\| \left\|Au'\right\|_{L_2} + \left\|B_2\right\| \left\|Au^2\right\|_{L_2}.$$

Using lemma 2 we obtain

$$\left\|P_1P_0^{-1}v\right\|_{L_2} = \left(c_1 \left\|B_1\right\| + c_0 \left\|B_2\right\|\right) \left\|P_0u\right\|_{L_2} = \alpha \left\|v\right\|_{L_2}.$$

Therefore the operator $E + P_1P_0^{-1}$ is inversible in $L_2\left((0,1)\,;H\right)$ and

$$u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1} f.$$

It is easy to see, that

$$\|u\|_{W_2^2((0,1);H)} < const\,\|f\|_{L_2((0,1);H)}\,.$$

The theorem is proved.

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