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ON UNIQUENESS OF SOLUTIONS OF CAUCHY TYPE PROBLEM FOR UNCOUNTABLE NORMAL SYSTEMS WITH FIRST ORDER PARTIAL DERIVATIVES

Abstract

In the paper the especially generalized neighborhood is introduced, three theorems of uniqueness of Tichonov type of solution of Cauchy's type problem for uncountable normal systems of first order partial differential equations are established and proved.

In the paper [1] the theorem of existence of solution of Cauchy problem for infinite systems of ordinary differential equations with infinite number of unknowns under the conditions of continuity of right-hand sides was proved by A.N.Tichonov. At the same time for infinite systems under additional Lipschitz type condition the uniqueness of the solution of Cauchy problem is established there.

On the other hand, in the paper [2], Tichonov, using his fixed point principle establishes the existence of solution of Cauchy problem for uncountable systems of ordinary differential equations. The results of the paper [2] are stated also in V.V.Nemytskiy's paper [3]. Indicating the generalization of the paper [1] in case when the system contains the unknown functions of two or more independent variables and their partial derivatives, in [4] the Lipschitz condition of right-hand sides of the system of the paper [1] without indication of possibility of its generalization, and the list of existing literature, which doesn't contain the suggested here results for uncountable systems with the partial derivatives, is given.

In this previous paper [5], the author generalizing Tichonov's result for uncountable systems of first order ordinary differential equations, established the existence of solution of Cauchy's type problem for normal uncountable uncountably dimensional systems of equations with first order partial derivatives. The following Cauchy's type problem was considered.

$$\frac{\partial y_{\alpha\beta}}{\partial t_\beta} = f_{\alpha\beta}\left(..., t_k, ..., x_j, ..., y_{sp}, ...\right), \tag{1}$$

where $\alpha, \beta, \gamma, s, p, k$ run the same values as $\alpha,\ 0 \leq \alpha \leq 1$, independently of each other, with initial data

$$y_{\alpha\beta}|_{t_\beta = t^0_\beta} = y^0_{\alpha\beta} = y^0_{\alpha\beta}\left(t'_\beta, x_\gamma\right), \tag{2}$$

where t'_β is a totality of the rest variables of type $t_{s\neq\beta},\ x_\gamma$ is a totality of the variables x_γ of the parameters x_γ - under the conditions of continuity of the right -hand sides $f_{\alpha\beta}$ of system (1) and initial data (2).

Then we'll use such brief denotation.

In the given paper, generalizing the condition of the paper [1] to uncountable systems, the author uses it for generalization of result [1] on a uniqueness of solution on normal uncountable uncountably dimensional systems with first order partial derivatives.

The suggested here result on uniqueness of solution of problem (1), (2) of Cauchy type, theorems 1, 2, 3 will be new not only for uncountable partial systems, but also for uncountable one-dimensional systems of ordinary differential equations of Tichonov's paper [2].

Definition 1. *We'll say, that the continual (uncountable) system of the functions $\{f_{\alpha\beta}\}$ of system (1) satisfies the continual Lipschitz condition of Tichonov's type, if there exist no more than countable (may be intersecting) subsystems $\{y_{mn}\}$, corresponding to $f_{\alpha\beta} \equiv f_{mn}$, dividing $\{f_{\alpha\beta}\}$ and indices $\{\alpha\beta\}$ on (may be interesting) classes $\{f_{mn}\} \rightleftarrows \{mn\}$, mutually corresponding, and for each subsystem $\{mn\}$ and $\forall f_{mn}$ own constants $K_{mn,\bar{m}\bar{n}} \geq 0$ are such, that for each $f_{\alpha\beta} \equiv f_{mn}$ from the given class $\{mn\}$ the following inequalities are fulfilled:*

$$\begin{aligned} &\left| f_{mn}\left(t_\beta, x_\gamma, ... y_{\alpha\beta\neq\bar{m}\bar{n}}, ..., y'_{\bar{m}\bar{n}}|_{\forall\bar{m}\bar{n}\in\{mn\}}, ...\right) - \right. \\ &\left. - f_{mn}\left(t_\beta, x_\gamma, ... y_{\alpha\beta\neq\bar{m}\bar{n}}, ..., y''_{\bar{m}\bar{n}}|_{\forall\bar{m}\bar{n}\in\{mn\}}, ...\right) \right| \leq \\ &\leq \sum_{\bar{m}\bar{n}\in\{mn\}} K_{mn,\bar{m}\bar{n}} \left| y'_{\bar{m}\bar{n}} - y''_{\bar{m}\bar{n}} \right| , \end{aligned} \tag{3}$$

where $\bar{m}\bar{n} \in \{mn\}$ means, that $\bar{m}\bar{n}$ runs all the values from the class $\{mn\}$.

The following theorems 1, 2, 3 show, that the second solution by virtue of these theorems is not contained in any of possible, defined there classes.

Theorem 1 (uniqueness of Tichonov type). *Let the right-hand sides of system (1) satisfy the*

1) continual Lipschitz condition of Tichonov type (3), moreover,

2) the sums $\sum\limits_{\{mn\}} K_{mn,\bar{m}\bar{n}} = A_{\bar{m}\bar{n}} < \infty$ converge and be uniformly bounded, i.e. on $\forall$ class $\{mn\}$

$$A_{\bar{m}\bar{n}} \leq A_{\{mn\}} < \infty \ , \quad A_{\{mn\}} = const,$$

at any numbering of the class $\{mn\}$ (or at least at any, that is equivalent).

Then there may exist only one system of solutions, for which the sum

$$\sum_{mn\in\{mn\}} |y_{mn}\left(x_\gamma, t_\beta\right)| < B_{\{mn\}} < \infty \ ^1 \quad for \quad \forall \{mn\} \tag{4}$$

(uniformly bounded on the subclasses $\{y_{mn}\}$) and satisfies the given initial data (2).

[1]A.N.Tichonov in [1], citing the similar conditions in case of countable systems of ordinary differential equations, underlined the necessity of similar condition in order that inequality (3) have sense and marks, that the sense is not the only matter: this condition is necessary in order that uniqueness not to be disturbed. In reality, this condition is sufficient, but is not necessary. In the following theorems 2 and 3 we'll show that the classes of uniqueness of the solution of the considered problems can be given in more general form, representing by itself new classes of uniqueness and that, namely, the requirement of sense for inequality (3) defines the most general widest classes of uniqueness, that just this requirement is not only necessary, but also a sufficient condition for definition of a class of uniqueness neighborhoods.

This is sufficient for uniqueness of solution, but this is also necessary for existence of sense of condition-inequality (3).

Proof of theorem 1. Similarly to [1] we suggest the presence of two systems of solutions: $\left\{y'_{\alpha\beta}\right\}$ and $\left\{y''_{\alpha\beta}\right\}$. Then as $\alpha\beta \in \{mn\}$, we have

$$\left|y'_{\alpha\beta} - y''_{\alpha\beta}\right| \leq \int\limits_{t_\beta^0}^{t_\beta} \left|\frac{\partial\left(y'_{\alpha\beta} - y''_{\alpha\beta}\right)}{\partial t_\beta}\right| dt_\beta \leq$$

$$\leq \int\limits_{t_\beta^0}^{t_\beta} \left|f_{mn}\left(x_\gamma, t_\beta, ..., y'_{\bar m\bar n}|_{\bar m\bar n \in \{mn\}}, y'_{\alpha\beta \neq \bar m\bar n}, ...\right) -\right.$$

$$\left.-f_{mn}\left(..., x_\gamma, t_\beta, ..., y''_{\bar m\bar n}|_{\bar m\bar n \in \{mn\}}, y''_{\alpha\beta \neq \bar m\bar n}, ...\right)\right| dt_\beta \leq$$

$$\leq \int\limits_{t_\beta^0}^{t_\beta} \sum_{\bar m\bar n \in \{mn\}} K_{mn,\bar m\bar n}\left|y'_{\bar m\bar n} - y''_{\bar m\bar n}\right|\ dt_\beta.$$

Numbering $\{mn\}$, $\ i = mn,\ \ j = \bar m\bar n$, similarly to [1] we have

$$Y_N = \sum_1^N \left|y'_i - y''_i\right| \leq \int\limits_{t_\beta^0}^{t_\beta} \sum_{i=1}^{\infty}\left\{\sum_{j=1}^{\infty} K_{ij}\left|y'_{\bar m\bar n} - y''_{\bar m\bar n}\right|\right\} dt_\beta \leq$$

$$\leq \int\limits_{t_\beta^0}^{t_\beta} \sum_{\{\bar m\bar n\}}\left\{\sum_{\{mn\}} K_{mn,\bar m\bar n}\left|y'_{\bar m\bar n} - y''_{\bar m\bar n}\right|\right\}\ dt_\beta \leq$$

$$\leq \int\limits_{t_\beta^0}^{t_\beta} A \cdot \sum_{\{\bar m\bar n\}}\left|y'_{\bar m\bar n} - y''_{\bar m\bar n}\right|\ dt_\beta \leq 2A \cdot B\left(t_\beta - t_\beta^0\right),$$

where $A = A_{\{mn\}},\ \ B = B_{\{mn\}}$.

Hence, it follows that the increasing sequence Y_N is uniformly bounded and its derivative numbers, as it is easy to see, are less than $2AB$.

This sequence converges to positive function $Y(x)$ for which $Y(x) \leq \int\limits_{t_\beta^0}^{t_\beta} A \cdot Y(x)\, dt_\beta$.

Hence, it follows that $Y(\cdot) \equiv 0,\ \ y'_{mn} \equiv y''_{mn}$ in the interval $0 \leq t_\beta - t_\beta^0 \leq \dfrac{1}{2A}$ [1].

The theorem is proved.

Theorem 2 (a generalized Tichonov type uniqueness). *Let the right-hand sides of system (1) satisfy the*

1) continual Lipschitz condition of Tichonov type (3), moreover, for fixed constants $K_{mn} \geq 0$

2) the sums $\sum\limits_{\{mn\}} K_{mn}K_{mn,\bar{m}\bar{n}} = A_{\bar{m}\bar{n}} < \infty$ *converge on* $\forall$ *class at any numbering of the class* $\{mn\}$ *and be uniformly bounded proportional to* $K_{\bar{m}\bar{n}}$*, i.e. on* $\forall$ *class* $\{mn\}$

$$A_{\bar{m}\bar{n}} \le A_{\{mn\}} \cdot K_{\bar{m}\bar{n}},\ A_{\{mn\}} = const\ .$$

Then there may exist only one system of solutions, for which the sum

$$\sum_{\bar{m}\bar{n}\in\{mn\}} K_{\bar{m}\bar{n}}|y_{\bar{m}\bar{n}}| \le B_{\{mn\}}, \tag{5}$$

where $B_{\{mn\}} = const$ *for* $\forall$ *class* $\{mn\}$ *and which satisfy the initial data (2).*

In particular, at $K_{mn} \equiv 1$ for $\forall mn$, theorem 1 is obtained.

Proof of theorem 2. Similarly to previous proof of theorem 1 for $\alpha\beta \in \{mn\}$ we have

$$\left|y'_{mn} - y''_{mn}\right| \le \int_{t_\beta^0}^{t_\beta} \left|\frac{\partial\left(y'_{mn} - y''_{mn}\right)}{\partial t_\beta}\right| dt_\beta \le$$

(as for f_{mn} we have the defined $K_{mn,\bar{m}\bar{n}} \ge 0$, mn is fixed, we have)

$$\le \int_{t_\beta^0}^{t_\beta} \sum_{\bar{m}\bar{n}\in\{mn\}} K_{mn,\bar{m}\bar{n}}|y'_{\bar{m}\bar{n}} - y''_{\bar{m}\bar{n}}|dt_\beta\ . \tag{6}$$

We'll renumber the class $\{mn\}$. Then to each mn it will correspond $\bar{m}\bar{n}$ with corresponding coefficients $K_{mn,\bar{m}\bar{n}} \ge 0$ one to one in sense of number.

Multiplying both sides of the last inequality by corresponding K_{mn} and summing, by virtue of conditions of theorem from (5) and (6) we obtain

$$Y_N(\cdot) = \sum_{\{mn\}} K_{mn}|y'_{mn} - y''_{mn}| \le$$

$$\le \int_{t_\beta^0}^{t_\beta} \sum_{\{\bar{m}\bar{n}\}} \sum_{\{mn\}} K_{mn}K_{mn,\bar{m}\bar{n}}|y'_{\bar{m}\bar{n}} - y''_{\bar{m}\bar{n}}|dt_\beta \le$$

$$\le 2A_{\{mn\}} \cdot B_{\{mn\}}\left(t_\beta - t_\beta^0\right)\ . \tag{7}$$

Hence, it follows, that $Y_N(\cdot)$ is a similarly continuous function ≥ 0, its derivative numbers are bounded and, hence, in the limit we have

$$Y(\cdot) \le \int_{t_\beta^0}^{t_\beta} A_{\{mn\}}Y(\cdot)\,dt_\beta\ .$$

Whence, it follows, that

$$\sum K_{\bar{m}\bar{n}}\left|y'_{\bar{m}\bar{n}} - y''_{\bar{m}\bar{n}}\right| \equiv 0,\quad i.e.\ \ y'_{\bar{m}\bar{n}} \equiv y''_{\bar{m}\bar{n}}\ .$$

(Q.E.D.)

Definition 2. *We'll call as especially generalized neighborhood of the function $\bar{y}_{\alpha\beta}$, all functions $y_{\alpha\beta}$ of the considered class for which*

$$\sum_{\{mn\}} K_{mn} \left| \bar{y}_{mn} - y_{mn} \right| \leq B_{\{mn\}}, \tag{8}$$

$B_{\{mn\}} = const$ *for* $\forall$ *class* $\{mn\}$, $K_{mn} \geq 0$ *are fixed constants.*

Theorem 3 (on the maximum generalized Tichonov's type uniqueness). *Let the right-hand sides of system (1) satisfy the*

1) continual Lipschitz condition of Tichonov's type (3), moreover

2) the sums

$$\sum_{\{mn\}} K_{mn} K_{mn,\bar{m}\bar{n}} = A_{\bar{m}\bar{n}} < \infty$$

converge on $\forall$ class at any numbering of the class $\{mn\}$, and be uniformly bounded proportional to $K_{\bar{m}\bar{n}}$, i.e. on $\forall$ class of $\{mn\}$

$$A_{\bar{m}\bar{n}} \leq A_{\{mn\}} K_{\bar{m}\bar{n}}, \quad A_{\{mn\}} = const \ .$$

Then at any especially generalized neighbourhood of the functions of the functions there can exist only one solution of Cauchy type problem (1), (2).

In particular, this theorem includes theorems 1 and 2.

Proof of theorem 3. It is done similarly to the proof of theorem 2 with some changes. Similarly for $\alpha\beta \in \{mn\}$ we estimate the difference $|y'_{mn} - y''_{mn}|$ for two solutions y'_{mn} and y''_{mn}. As a result we'll obtain inequality (6). By virtue of inequalities and conditions (6), (7), (8) the derivative numbers $Y_N(\cdot)$ are bounded and positive, the sequence of the continuous functions Y_N uniformly converges to the continuous functions, satisfies the inequalities

$$Y_N \leq \int_{t_\beta^0}^{t_\beta} A_{\{mn\}} Y dt_\beta \ .$$

Whence, it similarly follows the inequality

$$Y \leq \int_{t_\beta^0}^{t_\beta} A_{\{mn\}} Y dt_\beta,$$

hence, the fulfilment

$$\sum_{\{mn\}} K_{mn} \left| y'_{mn} - y''_{mn} \right| \equiv 0 \quad and \quad y'_{mn} \equiv y''_{mn}$$

The theorem is proved.

Consider, what relation is between neighbourhoods of uniqueness and conditions of the problems and uniqueness theorem. If we have especially generalized neighbourhood, then we can think, that the requirement of corresponding Lipschitz conditions with the coefficients $\{K_{ij}\}$ by any way corresponds to it. However, it is possible, that the right-hand side of the Lipschitz conditions may not correspond to especially generalized neighbourhood. Substitution of unknowns $y = K_{ij}u$ can lead to Lipschitz condition with $K_{ij} = 1$, but with substitution of f_{ij} by $f_{ij}\left(y/K_{ps}\right)$, and the coefficients of proportionality λ_{ij} may be taken as characteristic f, i.e. $\lambda_{ij} f_{ij}\left(y/K_{ps}\right)$, as the function will be another.

So, the coefficients of the Lipschitz conditions are by itself, and especially generalized neighbourhoods exist by itself. In other words, if in the given especially generalized neighbourhood for $f_{\alpha\beta}$ the definite Lipschitz conditions and other conditions of theorem, connected with coefficients of especially generalized neighbourhood and coefficients of Lipschitz conditions are fulfilled, the proved theorems of uniqueness are fulfilled.

As possibly,$K_i' \leq K_i''$, then especially generalized neighbourhoods can be assumed not only embedded to each other, but also intersected by different ways.

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