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ON INVERSE PROBLEM FOR SINGULAR STURM-LIOUVILLE OPERATOR FROM TWO SPECTRA

Abstract

In the paper an inverse problem by two given spectrum for second order differential operator with singularity of type $\frac{2}{r}+\frac{\ell(\ell+1)}{r^2}$ *in zero point (where* l *is a positive integer or zero), is studied. It is well known that the two spectrum* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *uniquely determine the potential function* $q(r)$ *in a singular Sturm-Liouville equation defined on interval* $(0,\pi]$.

One of the aims of the paper is to prove the generalized degeneracy of the kernel of integral equation in inverse problem. In particular we obtain a new proof of Hochstadt's theorem concerning the structure of the difference $\tilde{q}(r)-q(r)$.

1. Introduction. We will consider the equation

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{\ell(\ell+1)}{r^2}R+\left(E+\frac{2}{r}\right)R=0 \qquad (0<r<\infty). \tag{1}$$

In quantum mechanics the study of the energy levels of the hydrogen atom leads to this equation [2]. Substitution $R=y/r$ reduces equation (1) to the form

$$\frac{d^2y}{dr^2}+\left\{E+\frac{2}{r}-\frac{\ell(\ell+1)}{r^2}\right\}y=0. \tag{2}$$

Just as in the case of Bessel's equation, one can show that in a finite interval $[0,b]$ the spectrum is discrete.

As known [6], [11], for a solution of (2) which is bounded at zero one has the following asymptotic formula for $\lambda\to\infty$ $(E=\lambda)$:

$$\varphi(r,\lambda)=\frac{e^{\frac{\pi}{2\sqrt{\lambda}}}}{\left|\Gamma\left(\ell+1+\frac{i}{\sqrt{\lambda}}\right)\right|}\frac{1}{\sqrt{\lambda}}\cos\left[\sqrt{\lambda}r+\frac{1}{\sqrt{\lambda}}\ln\sqrt{\lambda}r-(\ell+1)\frac{\pi}{2}+\alpha\right]+o(1) \tag{3}$$

where $\alpha=\arg\Gamma\left(\ell+1+\frac{i}{\sqrt{\lambda}}\right)$.

We consider the two singular Sturm-Liouville problems

$$-y''+\left[\frac{\ell(\ell+1)}{r^2}-\frac{2}{r}+q(r)\right]y = \lambda y \quad (0<r\le\pi), \tag{4}$$

$$y(0) = 0, \tag{5}$$

$$y'(\pi)+Hy(\pi) = 0, \tag{6}$$

$$-y''+\left[\frac{\ell(\ell+1)}{r^2}-\frac{2}{r}+\tilde{q}(r)\right]y=\lambda y\quad(0<r\le\pi),\tag{7}$$
$$y(0)=0,$$
$$y'(\pi)+\tilde{H}y(\pi)=0,\tag{8}$$

in which the functions $q(r)$ and $\tilde{q}(r)$ are assumed to be real valued and square integrable. H and $\tilde{H}$ are finite real numbers.

We denote the spectrum of the first problem by $\{\lambda_n\}_0^\infty$, and the spectrum of the second by $\left\{\tilde{\lambda}_n\right\}_0^\infty$.

Next, we denote by $\varphi(r,\lambda)$ the solution of (4), and we denote by $\tilde{\varphi}(r,\lambda)$ the solution of (7) satisfying the initial condition (5).

It is well known that there exist a function $K(r,s)$ such that

$$\tilde{\varphi}(r,\lambda)=\varphi(r,\lambda)+\int_0^r K(r,s)\varphi(s,\lambda)\,ds.\tag{9}$$

The function $K(r,s)$ satisfies the equation

$$\frac{\partial^2 K}{\partial r^2}-\left[\frac{2}{r}-\frac{\ell(\ell+1)}{r^2}+\tilde{q}(r)\right]K=\frac{\partial^2 K}{\partial s^2}-\left[\frac{2}{s}-\frac{\ell(\ell+1)}{s^2}+q(s)\right]K\tag{10}$$

and the conditions

$$K(r,r)=\frac{1}{2}\int_0^r\left[\tilde{q}(t)-q(t)\right]dt,\tag{11}$$
$$K(r,0)=0.\tag{12}$$

After the transformations

$$z=\frac{1}{4}(r+s)^2,\quad w=\frac{1}{4}(r-s)^2,\quad K(r,s)=(z-w)^{-\nu+\frac{1}{2}}u(z,w)$$

we obtain the following problem $(-\nu+\frac{1}{2}=\alpha)$:

$$\frac{\partial^2 u}{\partial z\partial w}-\frac{\alpha}{z-w}\frac{\partial u}{\partial z}+\frac{\alpha}{z-w}\frac{\partial u}{\partial w}=\frac{qu}{4\sqrt{zw}}+\frac{u}{\sqrt{z(z-w)}}$$
$$\frac{\partial u}{\partial z}+\frac{\alpha}{z}u=\frac{1}{4}q\left(\sqrt{z}\right)z^{\nu-1},\quad u(z,z-\delta)=0.$$

This problem can be solved by using the Riemann method [4], [5], [14].

We put

$$c_n=\int_0^\pi\varphi^2(r,\lambda_n)\,dr,\quad \tilde{c}_n=\int_0^\pi\tilde{\varphi}^2\left(r,\tilde{\lambda}_n\right)dr$$

$$\rho(\lambda)=\sum_{\lambda_n<\lambda}\frac{1}{c_n},\quad \tilde{\rho}(\lambda)=\sum_{\tilde{\lambda}_n<\lambda}\frac{1}{\tilde{c}_n}.$$

The function $\rho(\lambda)$ $(\tilde{\rho}(\lambda))$ is called the spectral function of the problem (4)-(6) [(7)-(8)]. Problem (4)-(6) will be regarded as an unperturbed problem, while (7)-(8) will be considered to be a perturbation of (4)-(6).

It is a known [1] fact that knowledge of two spectrum for a given singular Sturm-Liouville equation makes it possible to recover its spectral function, i.e., to find the numbers $\{c_n\}$. More exactly, suppose that in addition to the spectrum of the problem (4)-(6), we also know the spectrum $\{\mu_n\}$ of the problem

$$-y'' + \left[\frac{\ell(\ell+1)}{r^2} - \frac{2}{r} + q(r)\right] y = \lambda y \quad ,$$

$$y(0) = 0, \quad y'(\pi) + H_1 y(\pi) = 0 \quad (H_1 \neq H). \tag{13}$$

Knowing $\{\lambda_n\}$ and $\{\mu_n\}$, we can calculate the numbers $\{c_n\}$. Similarly, for (7), if besides $\left\{\tilde{\lambda}_n\right\}$ we also know the spectrum $\{\tilde{\mu}_n\}$ determined by the boundary conditions

$$y(0) = 0, \quad y'(\pi) + \tilde{H}_1 y(\pi) = 0 \quad \left(\tilde{H}_1 \neq \tilde{H}\right), \tag{14}$$

it then follows that we can determine the numbers $\{\tilde{c}_n\}$.

It is also shown that

$$\sqrt{\lambda_n} = [n + \ell/2] + \frac{1}{\pi}\frac{\ln(n+\ell/2)}{n+\ell/2} + O\left(\frac{1}{n^2}\right),$$

$$\|\varphi_n\|^2 = \int_0^\pi \varphi_n^2(r)\,dr = \frac{\pi}{2} + \frac{\pi^2}{2}\frac{1}{n+\ell/2} + O\left(\frac{\ln n}{n^2}\right).$$

2. Statement of Results

This section will be devoted to a statement of the theorems, whose proofs will be given in a subsequent section.

Theorem 1. Consider the operator

$$Ly = -y'' + \left[\frac{\ell(\ell+1)}{r^2} - \frac{2}{r} + q(r)\right] y, \tag{15}$$

subject to boundary conditions

$$y(0) = 0 \tag{16}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{17}$$

where q is square integrable on $(0,\pi]$. Let $\{\lambda_n\}$ be the spectrum of L subject to (16) and (17).

If (17) is replaced by a new boundary condition,

$$y'(\pi) + H_1 y(\pi) = 0, \tag{18}$$

a new operator and a new spectrum, say $\{\mu_n\}$, result.

Consider now a second operator

$$\tilde{L}y = -y'' + \left[\frac{\ell(\ell+1)}{r^2} - \frac{2}{r} + \tilde{q}(r)\right] y, \tag{19}$$

where $\tilde{q}$ is square integrable on $(0,\pi]$. Suppose that, under the boundary conditions (16) and

$$y'(\pi) + \tilde{H}y(\pi) = 0, \tag{20}$$

$\tilde{L}$ has the spectrum $\left\{\tilde{\lambda}_n\right\}$, with $\tilde{\lambda}_n = \lambda_n$ for all n. $\tilde{L}$ with a boundary conditions (16) and

$$y'(\pi) + \tilde{H}_1 y(\pi) = 0 \tag{21}$$

is assumed to have the spectrum $\{\tilde{\mu}_n\}$.We assume that H, $H_1 \neq H$, $\tilde{H}$ and $\tilde{H}_1 \neq \tilde{H}$ are real numbers which are not infinite.

We shall denote by Λ_0 the finite index set for which $\tilde{\mu}_n \neq \mu_n$ and by Λ the infinite index set for which $\tilde{\mu}_n = \mu_n$. Under the about assumptions, it follows that the kernel $K(r,s)$ is degenerate in the extended sense

$$K(r,s) = \sum_{\Lambda_0} c_n \tilde{\phi}_n(r)\varphi_n(s)$$

where φ_n , $\tilde{\phi}_n$ are suitable solutions of (4) and (7).

Theorem 2. If the spectrum $\{\lambda_n\}$ and $\left\{\tilde{\lambda}_n\right\}$ coincide, and if the spectrum $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ differ in a finite number of their terms, then the integral equation

$$K(r,s) + F(r,s) + \int_0^r K(r,t)F(t,s)\,dt = 0 \quad (0 < s \leq r \leq \pi), \tag{22}$$

is degenerate in the extended sense. In (22)

$$\begin{aligned} F(r,s) &= \int_0^\infty \varphi(r,\lambda)\varphi(s,\lambda)\,d_\lambda\{\tilde{\rho}(\lambda) - \rho(\lambda)\} \\ &= \sum_{n=0}^\infty \left\{\frac{1}{\tilde{c}_n}\varphi\left(r,\tilde{\lambda}_n\right)\varphi\left(s,\tilde{\lambda}_n\right) - \frac{1}{c_n}\varphi(r,\lambda_n)\varphi(s,\lambda_n)\right\}. \end{aligned} \tag{23}$$

Theorem 3. If the spectrum $\{\lambda_n\}$ and $\left\{\tilde{\lambda}_n\right\}$ coincide, and that $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ differ in a finite number of their terms, i.e., $\tilde{\mu}_n = \mu_n$ for $n \in \Lambda$. Then

$$\tilde{q}(r) - q(r) = \sum_{\Lambda_0} \tilde{c}_n \frac{d}{dr}\left(\tilde{\phi}_n \cdot \varphi_n\right),$$

where φ_n , $\tilde{\phi}_n$ are suitable solutions of (4) and (7).

3. Proof of Theorem 1. From (9) it follows that

$$\tilde{\varphi}'(r,\lambda)=\varphi'(r,\lambda)+K(r,r)\varphi(r,\lambda)+\int\limits_0^r\frac{\partial K}{\partial r}\varphi(s,\lambda)\,ds,$$

and

$$\tilde{\varphi}'(r,\lambda)+\tilde{H}\tilde{\varphi}(r,\lambda)=\varphi'(r,\lambda)+\tilde{H}\varphi(r,\lambda)+$$
$$+K(r,r)\varphi(r,\lambda)+\int\limits_0^r\left(\frac{\partial K}{\partial r}+\tilde{H}K\right)\varphi(s,\lambda)\,ds.$$

Putting $r=\pi$, $\lambda=\lambda_n$ into the last equation and using boundary conditions (17); (20), we obtain

$$\left(\tilde{H}-H\right)\varphi(\pi,\lambda_n)+K(\pi,\pi)\varphi(\pi,\lambda_n)+$$
$$+\int\limits_0^\pi\left(\frac{\partial K}{\partial r}+\tilde{H}K\right)_{r=\pi}\varphi(s,\lambda_n)\,ds=0. \qquad (24)$$

As $n\to\infty$, $\varphi(\pi,\lambda_n)\to o(1)$ the integral on the right tends to zero. Therefore, from (24) we get

$$K(\pi,\pi)=H-\tilde{H}, \qquad (25)$$
$$\int\limits_0^\pi\left(\frac{\partial K}{\partial r}+\tilde{H}K\right)_{r=\pi}\varphi(s,\lambda_n)\,ds=0\quad(n=0,1,...). \qquad (26)$$

Since the system of functions $\varphi(s,\lambda_n)$ is complete, it follows from the last equation that

$$\left(\frac{\partial K}{\partial r}+\tilde{H}K\right)_{r=\pi}=0\quad(0<s\le\pi). \qquad (27)$$

We now use the condition imposed on the second-named spectrum. Using (9) again, we obtain

$$\tilde{\varphi}'(r,\lambda)+\tilde{H}_1\tilde{\varphi}(r,\lambda)=$$
$$=\varphi'(r,\lambda)+\tilde{H}_1\varphi(r,\lambda)+K(r,r)\varphi(r,\lambda)+\int\limits_0^r\left(\frac{\partial K}{\partial r}+\tilde{H}_1K\right)\varphi(s,\lambda)\,ds. \qquad (28)$$

Putting $r=\pi$ and $\lambda=\mu_n$ $(n\in\Lambda)$ and using (18); (21), we obtain

$$\int\limits_0^\pi\left(\frac{\partial K}{\partial r}+\tilde{H}_1K\right)_{r=\pi}\varphi(s,\mu_n)\,ds+$$
$$+\left(\tilde{H}_1-H_1\right)\varphi(\pi,\mu_n)+K(\pi,\pi)\varphi(\pi,\mu_n)=0.$$

In the last equation, as $n \to \infty$, the left side tends to zero and $\varphi(\pi, \mu_n) \to o(1)$. Therefore

$$K(\pi,\pi) = H_1 - \tilde{H}_1, \tag{29}$$

$$\int_0^\pi \left(\frac{\partial K}{\partial r} + \tilde{H}_1 K\right)_{r=\pi} \varphi(s,\mu_n)\, ds = 0 \quad n \in \Lambda. \tag{30}$$

Comparing (25) and (29), we obtain $H - \tilde{H} = H_1 - \tilde{H}_1$. For $n \in \Lambda_0$ we obtain from (28) (for $r = \pi$ and $\lambda = \mu_n$)

$$\int_0^\pi \left(\frac{\partial K}{\partial r} + \tilde{H}_1 K\right)_{r=\pi} \varphi(s,\mu_n)\, ds = \tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1 \tilde{\varphi}(\pi,\mu_n). \tag{31}$$

From (30) and (31) it follows that

$$\left(\frac{\partial K}{\partial r} + \tilde{H}_1 K\right)_{r=\pi} = \sum_{\Lambda_0} \frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1 \tilde{\varphi}(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2} \varphi(s,\mu_n) \quad (0 < s \le \pi). \tag{32}$$

From (27) and (32) we derive

$$K(\pi,s) = \frac{1}{\tilde{H}_1 - \tilde{H}} \sum_{\Lambda_0} \frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1 \tilde{\varphi}(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2} \varphi(s,\mu_n), \tag{33}$$

$$\frac{\partial K(r,s)}{\partial r}\,|_{r=\pi} =$$

$$= -\frac{\tilde{H}}{\tilde{H}_1 - \tilde{H}} \sum_{\Lambda_0} \frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1 \tilde{\varphi}(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2} \varphi(s,\mu_n) \quad (0 < s \le \pi). \tag{34}$$

The function $K(r,s)$ satisfies (10). Therefore, from the initial conditions (33) and (34) it follows that in the triangle I we have

$$\begin{aligned} K(r,s) &= \frac{1}{\tilde{H}_1 - \tilde{H}} \sum_{\Lambda_0} \frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1 \tilde{\varphi}(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2} \\ &\quad \times \left[\tilde{c}(r,\mu_n) - \tilde{H}\tilde{s}(r,\mu_n)\right] \varphi(s,\mu_n), \end{aligned} \tag{35}$$

where $\tilde{c}(r,\lambda)$ and $\tilde{s}(r,\lambda)$ are solutions of (7) satisfying the initial conditions

$$\tilde{c}(\pi,\lambda) = \tilde{s}'(\pi,\lambda) = 1 \quad , \quad \tilde{c}'(\pi,\lambda) = \tilde{s}(\pi,\lambda) = 0.$$

The function $K(r,s)$ and the sum (35) satisfy (12);therefore in the triangle II they coincide; consequently, they coincide in the triangle III as solutions of (10) satisfying the same initial conditions on the line $r = \pi/2$, etc., i.e., $K(r,s)$ is expressed by (35) throughout the triangle $0 < s \le r \le \pi$, (see, [7], [10], [12]).

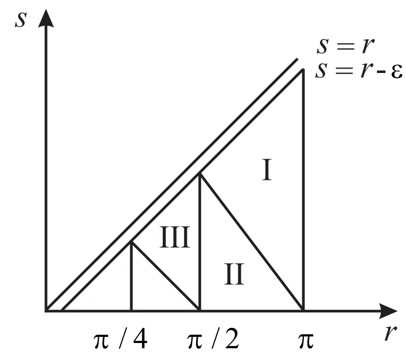

Fig.1

Proof of Theorem 2. From (22) we obtain, for $r=\pi$,

$$F(\pi,s)=-K(\pi,s)-\int_0^\pi K(\pi,t)\,F(t,s)\,dt.$$

Substituting in this equation, in place of $K(\pi,s)$, the expansion (33) and, in place of $F(t,s)$, the expansion (23) (in which λ_n and $\tilde{\lambda}_n$ are replaced by μ_n and $\tilde{\mu}_n$), we obtain, upon using the orthogonality of the functions $\varphi(s,\mu_n)$, the equality

$$F(\pi,s)=-\int_0^\pi \sum_{\Lambda_0}\alpha_n\varphi(t,\mu_n)\sum_{\Lambda_0}\frac{1}{\tilde{c}_k}\varphi(t,\tilde{\mu}_k)\,\varphi(s,\tilde{\mu}_k)\,dt,$$

where

$$\alpha_n=\frac{\tilde{\varphi}'(\pi,\mu_n)+\tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\left(\tilde{H}_1-\tilde{H}\right)c_n}$$

$$c_n=\|\varphi(s,\mu_n)\|^2 \quad , \quad \tilde{c}_k=\|\tilde{\varphi}(s,\tilde{\mu}_k)\|^2 .$$

Next, from (4) we easily obtain, for $k<N$

$$\int_0^\pi \varphi(t,\mu_n)\,\varphi(t,\tilde{\mu}_k)\,dt=\frac{\varphi'(\pi,\tilde{\mu}_k)\,\varphi(\pi,\mu_n)-\varphi'(\pi,\mu_n)\,\varphi(\pi,\tilde{\mu}_k)}{\mu_n-\tilde{\mu}_k} \qquad (\mu_n\neq\tilde{\mu}_k).$$

Therefore,

$$F(\pi,s)=\sum_{\Lambda_0}a_n\varphi(s,\tilde{\mu}_n), \tag{36}$$

where

$$a_n=\frac{1}{\tilde{c}_n}\left[\varphi(\pi,\tilde{\mu}_n)\sum_{\Lambda_0}\alpha_k\frac{\varphi'(\pi,\mu_k)}{\mu_k-\tilde{\mu}_n}-\varphi'(\pi,\tilde{\mu}_n)\sum_{\Lambda_0}\alpha_k\frac{\varphi(\pi,\mu_n)}{\mu_k-\tilde{\mu}_n}\right].$$

To calculate $F_r(\pi,s)$ we differentiate (22) with respect to r ; we then obtain

$$K_r + F_r + K(r,r)F(r,s) + \int\limits_0^r K_r(r,t)F(t,s)\,dt = 0.$$

Putting $r=\pi$ here and replacing $K(\pi,\pi)$ by (29), $F(\pi,s)$ by (36), and $K_r(\pi,s)$ by (34), we find that

$$F_r(\pi,s) = \sum_{\Lambda_0} b_n \varphi(s,\tilde{\mu}_n), \tag{37}$$

where the b_n are constants which we shall not write out.

From (23), $F(r,s)$ satisfies the equation

$$\frac{\partial^2 F}{\partial r^2} - \left[\frac{2}{r} - \frac{\ell(\ell+1)}{r^2} + \tilde{q}(r)\right]F = \frac{\partial^2 F}{\partial s^2} - \left[\frac{2}{s} - \frac{\ell(\ell+1)}{s^2} + q(s)\right]F.$$

Therefore, from the boundary conditions (36) and (37), we find that in the triangle I

$$F(r,s) = \sum_{\Lambda_0}\left[a_n c(r,\tilde{\mu}_n) + b_n s(r,\tilde{\mu}_n)\right]\varphi(s,\tilde{\mu}_n), \tag{38}$$

where $c(r,\lambda)$ and $s(r,\lambda)$ are solutions of (4) satisfying the boundary conditions

$$c(\pi,\lambda) = s'(\pi,\lambda) = 1 \quad , \quad c'(\pi,\lambda) = s(\pi,\lambda) = 0.$$

It is also evident from (23) that $F(r,s)$ satisfies the boundary condition

$$F(r,t)_{t=0} = 0.$$

This same boundary condition is satisfied, obviously, by the sum (38). Therefore, (38) is valid in the triangle II , etc., i.e., (38) holds throughout the triangle $0 < s \leq r \leq \pi$, i.e., the kernel $F(r,s)$ is degenerate in the extended sense, which is what we wished to prove.

Proof of Teorem 3. We obtain from (11)

$$\tilde{q}(r) - q(r) = 2\frac{dK(r,r)}{dr}.$$

Differentiating (35) and putting $s=r$, we obtain

$$\tilde{q}(r) - q(r) = \frac{2}{\tilde{H}_1 - \tilde{H}} \sum_{\Lambda_0} \frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2}$$

$$\times \frac{d}{dr}\left\{\left[\tilde{c}(r,\mu_n) - \tilde{H}\tilde{s}(r,\mu_n)\right]\varphi(\pi,\mu_n)\right\}. \tag{39}$$

Consequently

$$\tilde{q}(r) - q(r) = \sum_{\Lambda_0} \tilde{c}_n \frac{d}{dr}\left(\tilde{\phi}_n \cdot \varphi_n\right).$$

Where $\tilde{c}(r,\mu_n)-\tilde{H}\tilde{s}(r,\mu_n)=\tilde{\phi}_n$, $\varphi(r,\mu_n)=\varphi_n(r,\mu_n)$ and

$$\hat{c}_n=\frac{2\left[\tilde{\varphi}'(\pi,\mu_n)+\tilde{H}_1\tilde{\varphi}(\pi,\mu_n)\right]}{\left(\tilde{H}_1-\tilde{H}\right)\left\|\varphi(s,\mu_n)\right\|^2}.$$

This completes the proof of Theorem 3. We note that similar problems are investigated in [3], [9], [13].

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