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ON CLOSURE OF ALGEBRA OF PIECEWISE-CONTINUOUS FUNCTIONS

Abstract

In the paper the closure of any algebra of piecewise-continuous functions is described and analogy of Stone-Weierstrass theorem in the space of piecewise-continuous functions is obtained.

Stone-Weierstrass approximation theorem on closure of algebras in $C_R(K)$ (see [1], p.296) is well known. The similar question arises in studying the completeness of a system of eigen-functions of some discontinuous differential operators in algebra of piecewise-continuous functions. The similar directions are studied in the suggested paper.

First we introduce some denotation and notion that will be used in sequel.

Let $c \in (a,b)$ $(-\infty < a < b < \infty)$. By $C_R([a,b];c)$ we denote a space with sup norm of real functions f continuous on $[a,c] \cup (c,b]$ and having $f(c+0)$ finite right limits at the point c.

Similarly, let $a < c_1 < ... < c_n < b$ and $S = \{c_1, ..., c_n\}$ be the set of finite number of points; by $C_R([a,b];S)$ we denote a space with sup norm of real functions f continuous on $[a,,c_1] \cup (c_1,c_2] \cup ... \cup (c_n,b]$ and having right $f(c_i+0)$, $i = 1,2,...,n$ limits at the points $c_i, i = 1,2,...,n$.

Obviously, the spaces $C_R([a,b];c)$ and $C_R([a,b];S)$ are Banach spaces.

Let A be some sub-algebra of algebra $C_R([a,b];c)$. In $[a,b]$ we introduce equivalence relation in the following form:

$$x \sim y \overset{def}{\equiv} \forall f \in A : f(x) = f(y).$$

This relation decomposes the set $[a,b]$ into non-intersecting classes

$$\xi \equiv [x]_A = \{y \in [a,b] \,| \forall f \in A : f(y) = f(x)\}.$$

Denote by $\tilde{K}$ a set of equivalence classes $\xi = [x]_A$, and consider the projection function $p : [a,b] \to \tilde{K}$ defined by the equality $p(x) = [x]_A$. For any $f \in A$, respectively, on the set $\tilde{K}$ define the function $\tilde{f}$:

$$\tilde{f}(\xi) \equiv \tilde{f}([x]_A) = f(x) \qquad \left(\xi \in \tilde{K}\right). \tag{1}$$

Obviously $f = \tilde{f} \circ p$ and

$$\|f\|_{C_R(\tilde{K})} \equiv \sup_{\xi \in \tilde{K}} \left|\tilde{f}(\xi)\right| = \|f\|_{C_R(K)}.$$

By $[A]$ denote a set of functions $\tilde{f} : \tilde{K} \to R$ defined by equality (1). It is easily seen that the set $[A]$ forms algebra of functions and there is one-to-one correspondence between the algebras A and $[A]$.

Denote:

$$\xi_c \overset{def}{=} [c]_A = \{x \in |\forall f \in A : f(x) = f(c)\},$$

$$\xi_{c+0} \overset{def}{=} [x]_A^{c+0} = \{x\,|\forall f \in A : f(x) = f(c+0)\},$$
$$\xi_0 \overset{def}{=} [x]_A^0 = \{x\,|\forall f \in A : f(x) = 0\}.$$

Note that if $\xi \neq \xi_0$ and $\xi \neq \xi_{c+0}$, then the set $\xi = [x]_A \neq \varnothing$. In fact, therewith always $x \in x[x]_A$. In this case, it may happen that the set $\xi = [x]_A$ is one-element, i.e. $\xi = [x]_A = \{x\}$.

Note some properties of classes ξ, ξ_{c+0} and ξ_0 and some related notions.

(A^0) $\xi_0 = \varnothing \iff \forall x \in [a,b], \exists f \in A : f(x) \neq 0$.

(B^0) $\exists h \in A : h(c+0) \neq 0$.

If the conditions A^0 and B^0 are fulfilled, then we'll say that algebra A doesn't vanish on the set $[a,b]$.

(C^0) $\forall h \in A : h(c+0) = 0$.

(D^0) $\xi_c = \xi_{c+0} \iff A \subset C_R[a,b]$ i.e. all the functions of algebra A are continuous.

(E^0) $\xi_c \neq \xi_{c+0} \iff \exists g \in A : g(c) \neq g(c+0)$ i.e. there exists a discontinuous function in algebra A.

(F^0) $\xi_{c+0} = \varnothing \iff \forall x \in [a,b], \exists f \in A : f(x) \neq f(c+0)$.

(K^0) If $\xi_{c+0} = \varnothing$, (condition F^0) and all the equivalence classes $\xi = [x]_A = \{x\}$ are one-element, then

K^0 a) $\forall x_1, x_2 \in [a,b], \exists g \in A : g(x_1) \neq g(x_2)$;

K^0 b) $\forall x \in [a,b], \exists g \in A : g(x) \neq g(c+0)$.

In this case we'll say that algebra A divides the points of the set $[a,b]$.

Lemma 1. *For any $\xi, \eta \in \tilde{K}$ $(\xi \neq \eta)$ there exists such $\tilde{f} \in [A]$ that $\tilde{f}(\xi) \neq \tilde{f}(\eta)$.*

Proof. Let $\exists \xi, \eta \in \tilde{K}$ $(\xi \neq \eta)$ be such that for $\forall \tilde{f} \in [A]$ it holds $\tilde{f}(\xi) = \tilde{f}(\eta)$. Then $\forall x \in \xi$ and $\forall y \in \eta$ and for $\forall f \in A$

$$f(x) = \tilde{f}([x]_A) = \tilde{f}(\xi) = \tilde{f}(\eta) = \tilde{f}([y]_A) = f(y),$$

And this means, that $x \sim y$ and $\xi = \eta$. The obtained contradiction proves the lemma.

Lemma 2. *If $\xi_{c+0} \neq \varnothing$ and $\xi_{c+0} \neq \xi_0$, then for any $\xi_1, \xi_2 \in \tilde{K} \backslash \{\xi_{c+0}, \xi_0\}$ $\{\xi_1 \neq \xi_2\}$ and any real numbers $c_1, c_2, d \in R$ there exists such a function $\tilde{f} \in [A]$ that $\tilde{f}(\xi_1) = c_1$, $\tilde{f}(\xi_2) = c_2$, $\tilde{f}(\xi_{c+0}) = d$.*

Proof. If there exist the functions

$\hat{u}_i, \hat{\nu}_i, \hat{w}_i \in [A]$, $i = \overline{1,2}$ such that

$\hat{u}_1(\xi_1) = 1$, $\hat{u}_1(\xi_2) = 0$; $\hat{u}_2(\xi_1) = 1$, $\hat{u}_2(\xi_{c+0}) = 0$;

$\hat{\nu}_1(\xi_1) = 0$, $\hat{\nu}_1(\xi_2) = 1$; $\hat{\nu}_2(\xi_2) = 1$, $\hat{\nu}_2(\xi_{c+0}) = 0$;

$\hat{w}_1(\xi_1) = 0$, $\hat{w}_1(\xi_{c+0}) = 1$; $\hat{w}_2(\xi_2) = 0$, $\hat{w}_2(\xi_{c+0}) = 1$,

then denoting $\tilde{u} = \hat{u}_1\hat{u}_2$, $\tilde{\nu} = \hat{\nu}_1\hat{\nu}_2$ and $\tilde{w}_2 = \hat{w}_1\hat{w}_2$, we have that the desired function will be $\tilde{f} = c_1\tilde{u}_1 + c_2\tilde{u}_2 + d\tilde{w}$.

Prove the existence of the function $w_1 \in [A]$. The existence of the functions $\hat{u}_i, \hat{\nu}_i\,(i = 1,2)$ and $\hat{w}_2$ are similarly proved.

Since $\xi_{c+0} \neq \xi_0$, then there exist such functions $\tilde{g}, \tilde{h} \in [A]$ that $\tilde{g}(\xi_{c+0}) \neq \tilde{g}(\xi_1)$ and $\tilde{h}(\xi_{c+0}) \neq 0$. Assuming $\tilde{w} = \tilde{g} + \lambda\tilde{h}\,(\lambda \in R)$, we choose the number λ as follows: if $\tilde{g}(\xi_{c+0}) \neq 0$, then $\lambda = 0$; if $\tilde{g}(\xi_{c+0}) = 0$, then the number λ is chosen from the conditions:

$$\tilde{w}(\xi_{c+0}) - \tilde{w}(\xi_1) = -\tilde{g}(\xi_1) + \lambda[\tilde{h}(\xi_{c+0}) - \tilde{h}(\xi_1)] \neq 0.$$

Consequently, for the functions $\tilde{w}$ the conditions $\tilde{w}\left(\xi_{c+0}\right) \neq 0$ and $\tilde{w}\left(\xi_{c+0}\right) \neq \tilde{w}\left(\xi_1\right)$ are fulfilled. Then the function

$$\tilde{w}_1(\xi) = \frac{\tilde{w}(\xi)}{\tilde{w}\left(\xi_{c+0}\right)} \frac{\tilde{w}(\xi) - \tilde{w}(\xi_1)}{\tilde{w}\left(\xi_{c+0}\right) - \tilde{w}(\xi_1)} \in [A]$$

satisfies the conditions $\tilde{w}\left(\xi_{c+0}\right) = 1,\ \tilde{w}(\xi_1) = 0$. Lemma 2 is proved.

Lemma 3. *If $\xi_{c+0} \neq \varnothing$ and $\xi_{c+0} = \xi_0$ then fore any $\xi_1, \xi_2 \in K \backslash \{\xi_{c+0}\}$ $(\xi_1 \neq \xi_2)$ and any real numbers $c_1, c_2 \in R$ there exists such a function $\tilde{f} \in [A]$ that $\tilde{f}(\xi_1) = c_1,\ \tilde{f}(\xi_2) = c_2$.*

Proof is similar to the one of lemma 2.

Introduce the following denotation:

$$C_R^A([a,b];c) = \{f \in C_R([a,b];c) \mid f|_{[x]_A} \equiv f(x)\},$$

where $f|_M$ is the contraction of the function $f \in C_R([a,b];c)$ on the set $M \subset [a,b]$. Further, if $\xi_0 = [x]_A^0 \neq \varnothing$, we assume

$$C_R^{A,0}([a,b];c) = \{f \in C_R([a,b];c) \big| \ f|_{[x]_A^0} \equiv 0\},$$

and if $\xi_0 = [x]_A^0 = \varnothing$ we'll assume that $C_R^{A,0}([a,b];c) = C_R^A([a,b];c)$.

If for $\forall h \in A,\ h(c+0) = 0$ (condition C^0), then we assume

$$C_R^{A,c+0}([a,b];c) = \{f \in C([a,b];c) \big| \ f(c+0) \equiv 0\},$$

and if there exists such a function $h \in A$, that $h(c+0) \neq 0$ (condition B^0), then we'll assume $C_R^{A,c+0}([a,b];c) = C_R^A([a,b];c)$. Further, assume

$$E_R^A([a,b];c) = C_R^A([a,b];c) \cap C_R^{A,0}([a,b];c) \cap C_R^{A,c+0}([a,b];c).$$

Lemma 4. *For any $f \in E_R^A([a,b];c)$ and any $x, y, \in [a,b]$ there exists such a function $h_{xy} \in A$ that*

$$h_{xy}(x) = f(x), \quad h_{xy}(y) = f(y), \quad h_{xy}(c+0) = f(c+0). \tag{2}$$

Proof. Consider the there cases:

$$\text{a) } \xi_{c+0} = \varnothing,\ \xi_{c+0} \neq \xi_0; \text{ b) } \xi_{c+0} \neq \varnothing,\ \xi_{c+0} = \xi_0; \quad \text{c) } \xi_{c+0} = \varnothing.$$

In case a) we apply lemma 2. Let $x \in \xi,\ y \in \eta$. Then, by lemma 2 $\exists \tilde{h}_{\xi\eta} \in [A]$ is such that $\tilde{h}_{\xi\eta}(\xi) = f(x),\ \tilde{h}_{\xi\eta}(\eta) = f(y)$ and $\tilde{h}_{\xi\eta}\left(\xi_{c+0}\right) = f(c+0)$ (in particular, if $\xi = \xi_0$ or $\eta = \xi_0$, then $\tilde{h}_{\xi\eta}(\xi) = f(x) = 0$ or $\tilde{h}_{\xi\eta}(\eta) = f(y) = 0$). Hence for the function $h_{xy} = \tilde{h}_{\xi\eta} \circ p \in A$ condition (2) is fulfilled.

b) in this case, if $x \in \xi$ and $y \in \eta$, then by lemma 3 there exists such a function $\tilde{h}_{\xi\eta} \in [A]$, that $\tilde{h}_{\xi\eta}(\xi) = f(x),\ \tilde{h}_{\xi\eta}(\eta) = f(y)$. It is clear that in this case $\tilde{h}_{\xi\eta}\left(\xi_{c+0}\right) = \tilde{h}_{\xi\eta}(\xi_0) = 0 = f(c+0)$. Then for the function $h_{xy} = \tilde{h}_{\xi\eta} \circ p \in A$ condition (2) is fulfilled.

c) In this case condition F^0 is fulfilled, i.e. $\forall x \in [a,b]$, $\exists g \in A$, such that $g(x) \neq g(c+0)$.

Consider the two cases: 1c) $\exists h \in A$ such that $h(c+0) \neq 0$ (condition B^0); 2c) $\forall f \in A;\ h(c+0) = 0$ (condition C^0)

Case 1c). Let $x \in \xi,\ y \in \eta\ \left(\xi, \eta \in \tilde{K}\right)$ and $\xi \neq \eta$. Consider the following possible variants:

$$1.1\text{c})\ \xi \neq \xi_0,\ \eta \neq \xi_0;\quad 2.1\text{c})\ \xi = \xi_0,\ \eta \neq \xi_0.$$

In case 1.1c) there exist such functions $h_i, g_i \in A,\ i = 1,2,3$ that

$h_1(x) \neq 0,\ h_2(y) \neq 0,\ h_3(c+0) \neq 0$ and $g_1(x) \neq g_1(y),\ g_2(x) \neq g_2(c+0)$, $g_3(y) \neq g_3(c+0)$.

Then by means of the method of the proof of lemma 2 we can show that there exist the functions $u_i, \nu_i, w_i \in A,\ i = \overline{1,2}$, such that

$$u_1(x) = 1,\ u_1(y) = 0;\ u_2(x) = 1,\ u_2(c+0) = 0; \tag{3}$$

$$\nu_1(x) = 0,\ \nu_1(y) = 1;\ \nu_2(y) = 1,\ \nu_2(c+0) = 0; \tag{4}$$

$$w_1(x) = 0,\ w_1(c+0) = 1;\ w_2(x) = 0,\ w_2(c+0) = 1. \tag{5}$$

Assuming $u = u_1 u_2,\ \nu = \nu_1 \nu_2,\ w = w_1 w_2$ it can be easily seen that the function $h_{xy}(t) = f(x)\,u(t) + f(y)\,\nu(t) + f(c+0)\,w(t)$ will be the desired function.

In case 2.1c) for $\forall g \in S$ and $\forall f \in E_R^A([a,b];c)$ it holds $f(x) = g(x) = 0\ \ (x \in \xi)$. Therefore, as in case 1.1c) we can prove the existence of the functions $\nu_1, \nu_2, w_1, w_2 \in A$ satisfying conditions (4-5). Then $h_{xy}(t) = f(y)\,\nu(t) + f(c+0)\,w(t)$, where $\nu = \nu_1 \nu_2,\ w = w_1 w_2$ will be the desired function.

We are also to note that at case 1c) $\xi = \eta$ (i.e. $f(x) = f(y)$) the proof is similar.

Case 2c). In this case $\forall h \in A$ and $\forall f \in E_R^A([a,b];c)$ the conditions $f(c+0) = h(c+0) = 0$ are fulfilled. Therefore we must construct the function $h_{xy} \in A$ such that $h_{xy}(x) = f(x)$ and $h_{xy}(y) = f(y)$.

If $x \in \xi,\ y \in \eta\ \left(\xi, \eta \in \tilde{K}\right)$ in this case considering possible variances

$$1.2.\text{c})\ \xi \neq \xi_0,\ \eta \neq \xi_0\ \ (\xi \neq \eta);\ 2.2\text{c})\ \xi = \xi_0,\ \eta \neq \xi_0;\ 3.2\text{c})\ \xi = \eta$$

and arguing similarly, we can easily see that the proof of the existence of the function h_{xy} differs very little from the previous case 1 c). Lemma 4 is proved.

Theorem 1. *Let A be some algebra of algebra $C_R([a,b];c)$. Then $\bar{A} = E_R^A([a,b];c)$, where $\bar{A}$ is closure A by the norm of the space $C_R([a,b];c)$.*

Proof. Following the proof of Stone-Weierstrass theorem (see [2], p.183) we can show that if $f \in A$, then $|f| \in \bar{A}$. Hence, if $f_1, ..., f_n \in \bar{A}$ that $\max\{f_1(x), ..., f_n(x)\} \in \bar{A}$ and $\min\{f_1(x), ..., f_n(x)\} \in \bar{A}$.

Let any $\varepsilon > 0$ and the function $f \in E_R^A([a,b];c)$ be given. Prove that $f \in \bar{A}$.

Since $A \subset \bar{A}$ then it follows from lemma 4 that for any $x, y \in [a,b]$ we can find such a function $h_{xy} \in \bar{A}$ that $h_{xy}(x) = f(x),\ h_{xy}(y) = f(y)$ and $h_{xy}(c+0) = f(c+0)$. Then there exists the vicinity U_{xy} of the point y such that for any $t \in U_{xy}$ it holds $h_{xy}(t) > f(t) - \varepsilon$. We fix x; then open sets U_{xy} considered at all $y \in [a,b]$, form a covering of the compact $[a,b]$. Then there exist a finite number of $y_1, y_2, ..., y_m \in [a,b]$ (here we assume $y_1 = c$), for which $[a,b] = \bigcup_{i=1}^{m} U_{xy_i}$ and at $t \in U_{xy_i}$ it holds $h_{xy_i} > f(t) - \varepsilon$. Consider the function $g_x(t) = \max\{h_{xy_1}(t), ..., h_{xy_m}(t)\} \in \bar{A}$.

Obviously, $g_x(x) = f(x), g_x(c+0) = f(c+0)$ and for any $t \in [a,b] g_x(t) > \max\{h_{xy_k}(t), k = \overline{1,m}\} > f(t) - \varepsilon$.

Continuing in a similar way, for the function $g_x(t)$ at all x we construct a system of neighborhoods of V_x, covering $[a,b]$ where the inequality $g_x(x) < f(t) + \varepsilon \ (t \in V_x)$ is fulfilled and using the compactness of the segment $[a,b]$ we choose a finite number of the functions $g_{x_1}(t), ..., g_{x_n}(t)$. Assuming $\varphi(t) = \min\{g_{x_1}(t), ..., g_x(t)\} \in \bar{A}$ we can easily show that at all $t \in [a,b]$ the inequality $f(t) - \varepsilon < \varphi(t) + \varepsilon$ is fulfilled. This means that $f \in \bar{A}$ and $\bar{A} = E_R^A([a,b];c)$. Theorem 1 is proved.

Note some corollaries from theorem 1 related with conditions $\left(A^0 - K^0\right)$.

Corollary 1. *If the conditions D^0, A^0 and K^0 are fulfilled for algebra A, then $\bar{A} = C_R[a,b]$.*

This is a classic Stone-Weierstrass theorem.

Corollary 2. *If the condition D^0 is fulfilled for algebra A, then*

$$\bar{A} = \{f \in C_R[a,b] \mid f\big|_{[x]_A} \equiv f(x)\} \cap \{f \in C_R[a,b] \,\Big|\, f\Big|_{[x]_A^0} \equiv 0\}.$$

This result was obtained in [3].

Corollary 3. *If the conditions A^0, B^0, K^0 are fulfilled for A, i.e. algebra A doesn't vanish on the set $[a,b]$ and separates the points $[a,b]$, then $\bar{A} = C_R[a,b]$ [4].*

Let $C_R^A([a,b];S) = \{f \in C_R([a,b];S) \mid f\big|_{[x]_A} \equiv f(x)\}$. If $[x]_A^0 = \varnothing$, then we assume

$$C_R^{A,0}([a,b];S) = \{f \in C_R([a,b];S) \mid f\big|_{[x]_A} \equiv 0\},$$

if $\xi_0 = [x]_A^0 = \varnothing$, then we'll assume that

$$C_R^{A,0}([a,b];S) = C_R^A([a,b];S).$$

Further, let for some $i_k \in \{\overline{1,n}\}$, $k = 1,2,...,m \quad (m < n)$ and for any $h \in A$, $h(c_{i_k} + 0) = 0$. Assume $I = \{i_k\}_{k=1}^m$ and

$$C_R^{A,I}([a,b];S) = \{f \in C_R([a,b];S) \mid f(c_{i_k} + 0) = 0, \ i_k \in I\}.$$

If $I = \varnothing$, we'll adopt $C_R^{A,I}([a,b];S) = C_R^A([a,b];S)$. Assume

$$E_R^A([a,b];S) = C_R^A([a,b];S) \cap C_R^{A,0}([a,b];S) \cap C_R^{A,I}([a,b];S).$$

The following theorem is proved in a similar way.

Theorem 2. *Let A be some subalgebra of algebra $C_R([a,b];S)$ then $\bar{A} = E_R^A([a,b];S)$, where $\bar{A}$ is a closure of A by the norm of the space $C_R([a,b];S)$.*

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