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INVARIANT DETERMINATION OF Φ-OPERATORS

Abstract

In the present paper the invariant form of Φ- *operators is applied to the covariant tensor field of* (o,p) *type is determined, where* $S \in T_q^1(M_n)$. *The condition of the tenzor defining the* $\tilde{\Phi}^S$*-operator is found. It is obvious that the local form of this operator coincides with the local form of the first Yano Ako operator under some conditions. Further the invariant form of* Φ^S*-operator applied to the* $(0,p)$ *type covariant tensor field of the generalized first kind of Yano Ako operaot is determined. It becomes evident that the condition needed for this operator to define the tensor of type* $(0,p+q)$ *coincides with the purity condition of covariance tensor.*

The invariant form of $\Phi-$operators is determined for some types of tensor fields in Yano-Ako paper [1].In this paper the operators Φ and $\widetilde{\Phi}$ that are applied to tensor fields of type $(0,p)$ are associated with the fixed structures $\varphi \in T_1^1(M_n)$ and $S \in T_2^1(M_n)$. It should be noted that the Φ^φ-operator is applied only to pure tensor fields $\omega \in T_p^0(M_n)$. Unlike the operator Φ^φ the operator $\tilde{\Phi}^S$ is applied to tensor fields $\omega \in T_p^0(M_n)$ under the conditions which are no longer the purity conditions of the tensor field ω relative to the structure S. In case of $\omega \in T_p^1(M_n)$ the $\tilde{\Phi}^S$ operator is determined and either the conditions of purity ω relative to S-structure, i.e., in this case we can apply the notation Φ^S.

In the represented paper the Yano-Ako results develop in two directions:

1) in case of $S \in T_q^1(M_n)$ and $\omega \in T_p^0(M_n)$ it was obtained the invariant form of Φ^S operator whose local form coincides with A.A.Salimov operator [2];

2) for $S \in T_q^1(M_n)$ and $\omega \in T_p^0(M_n)$ the operator $\tilde{\Phi}$ is invariantly determined.

Note that as a result of action of two operators Φ and $\tilde{\Phi}$ we obtain the tensor.

1. Operator $\tilde{\Phi}^S\omega$ where $S \in T_q^1(M_n)$, $\omega \in T_q^0(M_n)$. Let $S \in T_q^1(M_n)$ and $\omega \in T_p^0(M_n)$. Consider the following expression:

$$\left(L_{S(x_1,\dots,x_q)}\omega\right)(z_1,\dots,z_p) - \sum_{i=1}^{q}\left(L_{x_i}(\omega\ o\ S)\right)(x_1,\dots,x_{i-1},z_1,x_{i+1},\dots,x_q,z_2,\dots,z_p) +$$

$$+\sum_{i=k+1}^{q}\sum_{k=1}^{q-1}(\omega\ o\ S)(x_1,\dots,x_{k-1},z_1,x_{k+1},\dots,x_{i-1},[x_i,x_k],x_{i+1},\dots,x_q,z_2,\dots,z_p) =$$

$$= S(x_1,\dots,x_q)(\omega(z_1,\dots,z_p)) - \sum_{i=1}^{p}\omega(z_1,\dots,z_{i-1},[S(x_1,\dots,x_q),z_i],z_{i+1},\dots,z_p) -$$

$$-\sum_{i=1}^{q}x_i\left(\omega\left(S(x_1,\dots,x_{i-1},z_1,x_{i+1},\dots,x_q),z_2,\dots,z_p\right)\right) +$$

$$+\sum_{i=1}^{q}(\omega\ o\ S)(x_1,\dots,x_{i-1},[x_i,z_1],x_{i+1},\dots,x_q,z_2,\dots,z_p) +$$

$$+\sum_{i=2}^{p}\sum_{k=1}^{q}(\omega\ o\ S)(x_1,...,x_{k-1},z_1,x_{k+1},...,x_q,z_2,...,z_{i-1},[x_k,z_i],z_{i+1},...,z_p)+$$

$$+\sum_{k=2}^{q}\sum_{i=1}^{k-1}(\omega\ o\ S)(x_1,...,x_{i-1},[x_k,x_i],x_{i+1},...,x_{k-1},z_1,x_{k+1},...,x_q,z_2,...,z_p) \quad (1)$$

where $x_1,...,x_q,z_1,...,z_p\in T_0^1(M_n)$.

In case of $q=2,p=1$ expression (1) will have the following form [1]

$$\left(L_{S(x,y)}\omega\right)(z)-(L_x(\omega\ o\ S))(z,y)-(L_y(\omega\ o\ S))(x,z)+(\omega\ o\ S)([x,y],z)=$$
$$=(S(x,y))(\omega(z))-x(\omega(S(z,y)))-$$
$$-y(\omega(S(x,z)))-\omega(S([x,y],z))-\omega\left(\tilde{\Phi}(x,y)z\right), \quad (2)$$

where $x,y,z\in T_0^1(M_n)$, $(\omega\ o\ S)(x,y)=\omega(S(x,y))$, $\tilde{\Phi}(x,y)z=-(L_zS)(x,y)$.

Expression (2) determines the tensor of type $(0,3)$.

Note, that the local form of operator (2) has the following form [1]

$$S^a_{kj}\partial_a\omega_{i_p...i_1}-\partial_k\left(S^a_{ij}\omega_a\right)-\partial_j\left(S^a_{ki}\omega_a\right)+\partial_i\left(S^a_{kj}\omega_a\right).$$

We can see expression (1) will be linear by $z_1,...,z_p$. For checking the linearity by $x_1,...,x_q$ we check the linearity of expression (1) in such parts where the linearity by $x_1,...,x_q$ is broken:

$$-\omega(z_1,...,[S(f_1x_1,...,f_qx_q),z_i],...,z_p)+$$
$$+(\omega\ o\ S)(z_1,f_2x_2,...,f_qx_q,z_2,...,[f_1x_1,z_i],...,z_p)+...+$$
$$+...+(\omega\ o\ S)(f_1x_1,...,f_{q-1}x_{q-1},z_1,z_2,...,[f_qx_q,z_i],...,z_p)=$$
$$=f_1f_2...f_q\{-\omega(z_1,...,[S(x_1,...,x_q),z_i],...,z_p)\}+$$
$$+(\omega\ o\ S)(z_1,x_2,...,x_q,z_2,...,[x_1,z_i],...,z_p)+...+$$
$$+(\omega\ o\ S)(x_1,...,x_{q-1},z_1,z_2,...,[x_q,z_i],...,z_p)+$$
$$+z_i(f_1...f_q)\,\omega(z_1,...,S(x_1,...,x_q),...,z_p)-$$
$$-f_2...f_q(z_if_1)(\omega\ o\ S)(z_1,x_2,...,x_q,z_2,...,x_1,...,z_p)-...-$$
$$-...-f_1...f_{q-1}(z_if_q)(\omega\ o\ S)(x_1,...,x_{q-1},z_1,z_2,...,x_q,...,z_p),$$

where $f_1,f_2,...,f_q\in T_0^0(M_n)$, $i=\overline{1,p}$.

So, it holds

Theorem 1. *Let* $S\in T_q^1(M_n)$ *and* $\omega\in T_p^0(M_n)$. *If*

$$\begin{gathered}\omega(z_1,...,\underset{\text{the } i-th \text{ place}}{S(x_1,...,x_q)},...,z_p)=\omega(S(z_1,x_2,...,x_q),z_2,...,\underset{\text{the } i-th \text{ place}}{x_1},...,z_p)=\\ =...=\omega(S(x_1,x_2,...,x_{q-1},z_1),z_2,...,\underset{\text{the } i-th \text{ place}}{x_q},...,z_p),\quad(1\le i\le p),\end{gathered} \quad (3)$$

then the operator $\tilde{\Phi}^S\omega$ determines the tensor of type $(0,p+q)$.

Remark 1. The local form of the operator $\tilde{\Phi}^S\omega$ has the form [2]

$$S^h_{i_1...i_q}\partial_h\omega_{j_1...j_p}-\partial_{i_1}\left(\omega_{mj_2...j_p}S^m_{j_1i_2...i_q}\right)-...-\partial_{i_q}\left(\omega_{mj_2...j_p}S^m_{i_1...i_{q-1}j_1}\right)+$$

$$+\omega_{mj_2...j_p}\partial_{j_1}S^m_{i_1...i_q}+...+\omega_{j_1...j_{p-1}m}\partial_{j_p}S^m_{i_1...i_q},$$

where the tensor field ω satisfies the conditions

$$\omega_{j_1...m...j_p}S^m_{i_1...i_q}=\omega_{mj_2...i_1...j_p}S^m_{j_1i_2...i_q}=$$

$$=...=\omega_{mj_2...i_q...j_p}S^m_{i_1i_2...i_{q-1}j_1},\,(1\le i\le p)\,. \tag{4}$$

Condition (4) is a local form of condition (3).

2. Operator $\Phi^S\omega$ where $S\in T^1_q(M_n)$, $\omega\in T^0_p(M_n)$. Let $S\in T^1_q(M_n)$ and $\omega\in T^0_p(M_n)$. Consider the following expression:

$$\left(L_{S(x_1,...,x_q)}\omega\right)(z_1,...,z_p)-\sum_{i=1}^{q}\left(L_{x_i}\left(\omega\ o\ S\right)\right)(x_1,...,x_{i-1},z_1,x_{i+1},...,x_q,z_2,...,z_p)+$$

$$+\sum_{e=1}^{q}\sum_{k=2}^{p}\omega\left(z_1,...,z_{k-1},S\left(x_1,...,x_{e-1},L_{x_e}z_k,x_{e+1},...,x_q\right),z_{k+1},...,z_p\right)-$$

$$-\sum_{e=1}^{q}\sum_{k=2}^{p}\omega\left(S\left(x_1,...,x_{e-1},z_1,x_{e+1},...,x_q\right),z_2,...,z_{k-1},L_{x_e}z_k,z_{k+1},...,z_p\right)+$$

$$+\sum_{e=1}^{q-1}\sum_{k=e+1}^{p}\left(\omega\ o\ S\right)(x_1,...,x_{e-1},z_1,x_{e+1},...,x_{k-1},[x_k,x_e],x_{k+1},...,x_q,z_2,...,z_p)=$$

$$=S(x_1,...,x_q)\left(\omega\left(z_1,...,z_p\right)\right)-\sum_{i=1}^{q}x_i\left(\omega\left(S\left(x_1,...,x_{i-1},z_1,x_{i+1},...,x_q\right),z_2,...,z_p\right)\right)-$$

$$-\sum_{k=1}^{p}\omega\left(z_1,...,z_{k-1},\Phi^S\left(x_1,...,x_q\right)z_k,z_{k+1},...,z_p\right)+$$

$$+\sum_{e=1}^{q-1}\sum_{k=e+1}^{p}(\omega\circ S)(x_1,...,x_{e-1},z_1,x_{e+1},...,x_{k-1},[x_e,x_k],x_{k+1},...,x_q,z_2,...,z_p)\,, \tag{5}$$

where

$$x_1,x_2,...,x_q,z_1,z_2,...,z_p\in T^1_0(M_n)\,,\ \left(\omega\ o\ S\right)(z_1,x_2,...,x_q,z_2,...,z_p)=$$

$$=\omega\left(S\left(z_1,x_2,...,x_q\right),z_2,...,z_p\right)$$

and

$$\Phi^S\left(x_1,...,x_q\right)z_k\overset{def}{=}\left[S\left(x_1,...,x_q\right),z_k\right]-$$

$$-S\left([x_1,z_k],x_k,...,x_q\right)-...-S\left(x_1,x_2,...,x_{q-1},[x_q,z_k]\right),\ k=\overline{1,p}.$$

Assume that the Φ-operator is associated with affine structure φ. Application of the operator Φ to the field $\omega \in T_1^0(M_n)$ or in $p=1,\ q=1$ has the following form [1]

$$\left(L_{\varphi x}\omega - L_x(\omega \circ \varphi)\right)(y) = (\varphi x)(\omega(y)) - x(\omega(\varphi y)) - \omega(\Phi(x)\,y)\,. \tag{6}$$

In case $\omega \in T_2^0(M_n)$ the operator $\Phi^{\varphi}\omega$ has the form [1]

$$\left(L_{\varphi x}\omega - L_x(\omega \circ \varphi)\right)(y,z) + \omega(y,\ \varphi L_x z) - \omega(\varphi y, L_x z) =$$

$$= (\varphi x)(\omega(y,z)) - x(\omega(\varphi y, z)) - \omega(\Phi(x)\,y, z) - \omega(y, \Phi(x)\,z)\,. \tag{7}$$

Expression (7) determines the tensor of type $(0,3)$ if the purity condition

$$\omega(\varphi y, z) = \omega(y, \varphi z)\,. \tag{8}$$

is fulfilled.

The local expressions (6) and (7) will be

$$\varphi_j^a \partial_a \omega_i - \varphi_j^a \omega_a - \omega_a\left(\partial_j \varphi_i^a - \partial_i \varphi_j^a\right),$$

$$\varphi_k^a \partial_a \omega_{ji} - \partial_{[k}\left(\omega_{|a|i}\varphi_{j]}^a\right) + \omega_{ja}\partial_i \varphi_k^a + \omega_{ai}\partial_j \varphi_k^a,$$

respectively, and the local expressions (8) are the followings:

$$\omega_{ja}\varphi_i^a = \omega_{ai}\varphi_j^a.$$

It is evident that (5) will be linear by $x_1,...,x_p$. For checking the linearity of $z_1,...,z_p$ we'll check the linearity of expression (5) in that parts where the linearity by $z_1,...,z_p$ is broken:

$$S(x_1,...,x_q)(\omega(h_1 z_1,...,h_p z_p)) -$$

$$-\sum_{i=1}^{q} x_i\left(\omega\left(S(x_1,...,x_{i-1},h_1 z_1,x_{i+1},...,x_q),h_2 z_2,...,h_p z_p\right)\right) -$$

$$-\sum_{k=1}^{p}\omega\left(h_1 z_1,...,h_{k-1}z_{k-1},\Phi^S(x_1,...,x_q)\,h_k z_k,h_{k+1}z_{k+1},...,h_p z_p\right) +$$

$$\sum_{e=1}^{q-1}\sum_{k=e+1}^{p}(\omega\ o\ S)\left(x_1,...,x_{e-1},h_1 z_1,x_{e+1},...,x_{k-1},[x_e,x_k],x_{k+1},...,x_q,h_2 z_2,...,h_p z_p\right) =$$

$$= h_1...h_p\left\{S(x_1,...,x_q)(\omega(z_1,...,z_p)) -\right.$$

$$-\sum_{i=1}^{q} x_i\left(\omega\left(S(x_1,...,x_{i-1},z_1,x_{i+1},...,x_q),z_2,...,z_p\right)\right) -$$

$$-\sum_{k=1}^{p}\omega\left(z_1,...,z_{k-1},\Phi^S(x_1,...,x_q)\,z_k,z_{k+1},...,z_p\right) +$$

$$\left.+\sum_{e=1}^{q-1}\sum_{k=e+1}^{p}(\omega\ o\ S)\left(x_1,...,x_{e-1},z_1,x_{e+1},...,x_{k-1},[x_e,x_k],x_{k+1},...,x_q,z_2,...,z_p\right)\right\} -$$

$$-\sum_{k=1}^{p}\sum_{i=1}^{q} h_1 ... (x_i h_k) ... h_p \{\omega(S(x_1, ..., x_{i-1}, z_1, x_{i+1}, ..., x_q), z_2, ..., z_p) -$$

$$-\omega(z_1, ..., z_{k-1}, S(x_1, ..., x_{i-1}, z_k, x_{i+1}, ..., x_q), z_{k+1}, ..., z_p)\},$$

where $h_1, ..., h_p \in T_0^0(M_n)$.

And so it holds.

Theorem 2. *Let* $S \in T_q^1(M_n)$ *and* $\omega \in T_p^0(M_n)$.. *If*

$$\omega(S(x_1, ..., x_{i-1}, z_1, x_{i+1}, ..., x_q), z_2, ..., z_p) =$$

$$= \omega(z_1, ..., z_{k-1}, S(x_1, ..., x_{i-1}, z_k, x_{i+1}, ..., x_q), z_{k+1}, ..., z_p), \quad i = \overline{1, q}, \quad k = \overline{2, p},$$

where $x_1, ..., x_q, z_1, ..., z_p \in T_0^1(M_n)$ then the operator $\Phi^S\omega$ determines the tensor of type $(0, p+q)$.

Introduce the following notation:

$$\Phi^S\omega(x_1, ..., x_q, z_1, ..., z_p) \overset{def}{=} S(x_1, ..., x_q)(\omega(z_1, ..., z_p)) -$$

$$-\sum_{i=1}^{q} x_i(\omega(S(x_1, ..., x_{i-1}, z_1, x_{i+1}, ..., x_q), z_2, ..., z_p)) -$$

$$-\sum_{k=1}^{p} \omega\left(z_1, ..., z_{k-1}, \Phi^S(x_1, ..., x_q) z_k, z_{k+1}, ..., z_p\right) +$$

$$+\sum_{e=1}^{q}\sum_{k=e+1}^{p} (\omega \ o \ S)(x_1, ..., x_{e-1}, z_1, x_{e+1}, ..., x_{k-1}, [x_e, x_k], x_{k+1}, x_q, z_2, ..., z_p).$$

Remark 2. The local form of the operator $\Phi^S\omega$ has the form [1]

$$\Phi^S(\omega)_{i_1...i_q j_1...j_p} \overset{def}{=} S^h_{i_1...i_q}\partial_h\omega_{j_1...j_p} -$$

$$-\sum_{a=1}^{q}\partial_{i_a}\left(S^m_{i_1...i_{a-1}j_1 i_{a+1}...i_q}\omega_{m j_2...j_p}\right) + \sum_{b=1}^{p}\omega_{j_1...m...j_p}\partial_{j_b}S^m_{i_1...i_q},$$

where the tensor field ω satisfies the conditions

$$\omega_{m j_2...j_p} S^m_{i_1...i_{e-1}j_1 i_{e+1}...i_q} = \omega_{j_1...j_{k-1}m_{j_{k+1}}...j_p} S^m_{i_1...i_{e-1}j_k i_{e+1}...i_q}, \quad i = \overline{1, q}, \quad k = \overline{2, p}.$$

This is the purity condition of the tensor $\omega \in T_p^0(M_n)$.

References

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