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ON REMOVABLE SETS OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS OF THE SECOND ORDER

Abstract

In the paper Neumann problem is considered for nondivergent elliptic equations of the second order with minor terms and the sufficient removability condition of compact is proved.

Let's consider the following boundary value problem in the bounded domain $D \subset R^n,\ n \geq 3:$

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u = 0, \tag{1}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial D \backslash E} = 0, \tag{2}$$

where ∂D is boundary of the domain D, E is some compact set lying on ∂D, and $\frac{\partial}{\partial \nu}$ is a derivative by conormal. Call the set E removable relative to the Neumann problem for equation (1) in $C^{0,\lambda}(D),\ o < \lambda < 1$ if from

$$Lu = 0,\ x \in D \backslash E, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial D \backslash E} = 0, \quad u(x) \in C^{0,\lambda}(D), \tag{3}$$

it follows $u(x) \equiv 0$ in D.

Relative to the coefficients we assume the fulfilment of the following condition

$$\gamma |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \leq \gamma^{-1} |\xi|^2, \tag{4}$$

$$|a_{ij}(x) - a_{ij}(y)| \leq k_1 |x - y| \tag{5}$$

$$|b_i(x)| \leq b_0; \qquad -b_0 \leq c(x) \leq 0. \tag{6}$$

Here $i, j = \overline{1, n},\ k_1$ is a constant. Besides the minor coefficients are the functions measurable in D.

In the paper the sufficient conditions of removability of compact is proved relative to the problem (1)-(2) in the space $C^{0,\lambda}(D)$. The corresponding result was obtained by L.Carleson [1] for the Laplace equation. In case of Neumann problem the questions on removability for Laplace equation in piecewise-smooth domains are considered in [2], [3]. The questions of removability for solutions of the first boundary value problem for elliptic and parabolic equations are considered in [4]. The removability conditions of a compact in space of continuous functions are constructed in [5].

Denote by $B_R(z)$ and $S_R(z)$ the ball $\{x : |x-z| < R\}$ and the sphere $\{x : |x-z| = R\}$ of radius R with the center at the point $z \in R^n$. $\frac{\partial u}{\partial v}$ is a derivative by conormal determined by the equality

$$\frac{\partial u(x)}{\partial v} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \cos(n, x_j),$$

where $\cos(n, x_j)$, $j = \overline{1,n}$ are direction cosines of a unique external normal to the surface.

We'll denote by $m_H^s(A)$ the Hausdorff measure of the set A of order $s > 0$.

Further the notation $C(\cdot)$ means that the positive constant C depends only on the content of parantheses.

Let's fix an arbitrary $\varepsilon > 0$ and we cover the set E by the ball of redius r_i in that way

$$\sum_{i=1}^{m} r_i^{n-2+\alpha} < \varepsilon. \tag{7}$$

Consider the spheres $S_R(0)$ and $S_{2R}(0)$. Denote by Σ_i the surfaces separating the sphere of redius r_i from the sphere of redius $2r_i$ in domain D and separating the singular points ∂D like that

$$\int_{\Sigma_i} \left| \frac{\partial u}{\partial v} \right| ds \le C_1 \underset{r_i < r < 2r_i}{osc} u \cdot r_i^{n-2}, \tag{8}$$

where C_1 depends on γ and n. The existence of such surfaces follows from [6].

From the conditions on coefficients it follows that almost everywhere in D there exist bounded derivatives from a_{ij}. Not loosing generality we can assume that such derivatives exist everywhere in D.

Let D_Σ be an open set situated in $D \backslash E$, whose boundary consists of unification of Σ_i and Γ where $\Sigma = \bigcup_{k=1}^{\infty} \Sigma_k$, $\Gamma = \partial D \backslash \bigcup_{k=1}^{\infty} D_k^+$, D_k^+ is a part of D_k remaining after the removing of points situated between Σ and $S_{2r_k}(x^k)$; $k = 1, 2, \ldots$. Denote by D'_Σ the arbitrary connected component D_Σ, and by B we denote the ellipitic operator of divergent structure

$$B = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right).$$

According to Green formula for any functions $z(x)$ and $W(x)$ pertaining to the intersection $C^2(D'_\Sigma) \cap C^1\left(\overline{D}'_\Sigma\right)$ we have

$$\int_{D_{\Sigma'}} (zBW - WBz)\, dx = \int_{D_{\Sigma'}} \left(z \frac{\partial W}{\partial \nu} - W \frac{\partial z}{\partial \nu} \right) ds. \tag{9}$$

As is known $u(x) \in C^1\left(\overline{D}_{\Sigma'}\right)$ (see [7]). From (9) choosing the functions z and W we have

$$\int_{D_{\Sigma'}} B\left(u^2\right) dx = 2 \int_{\partial D_{\Sigma'}} u \frac{\partial u}{\partial \nu} ds.$$

Since we consider the bounded solutions $|u(x)| \leq M < \infty,\ x \in \overline{D}$ then subject to (7) and

$$\int\limits_{D_{\Sigma'}} \left|\frac{\partial u}{\partial \nu}\right| ds \leq C_1 \cdot r_k^{n-2+\alpha}, \quad k = 1, 2, ... \tag{10}$$

we have

$$\int\limits_{D_{\Sigma'}} B\left(u^2\right) dx \leq 2Ma_0 \sum_{k=1}^{\infty} \int\limits_{\Sigma_k} \left|\frac{\partial u}{\partial \nu}\right| ds \leq 2Ma_0 C_1 \sum_{k=1}^{\infty} r_k^{n-2+\alpha} < C_2 \cdot \varepsilon. \tag{11}$$

Besides

$$Bu = Lu + \sum_{i=1}^{n} d_i(x) u_i - C(x) u,$$

where

$$d_i(x) = \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} - b_i(x),\ i = 1, ..., n.$$

Allowing for

$$B\left(u^2\right) = 2u \cdot Bu + 2 \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j},$$

and by virtue of conditions (4)-(6)

$$|d_i(x)| \leq d_0 < \infty; \quad i = 1, ...n,$$

from (11) we have

$$2 \int\limits_{D_{\Sigma'}} u \sum_{i=1}^{n} d_i(x) u_{x_i} dx - 2 \int\limits_{D_{\Sigma'}} u^2 c(x) dx + 2 \int\limits_{D_{\Sigma'}} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} dx < C_2 \cdot \varepsilon.$$

Hence for any $\alpha > 0$ applying Cauchy inequality we get

$$2\gamma \int\limits_{D_{\Sigma'}} |\nabla u|^2 dx < 2d_0 \int\limits_{D_{\Sigma'}} \sum_{i=1}^{n} |u| \cdot |u_{x_i}| dx + C_2 \cdot \varepsilon \leq d_0 \alpha \int\limits_{D_{\Sigma'}} |\nabla u|^2 dx +$$

$$+ \frac{d_0 n}{\alpha} \int\limits_{D_{\Sigma'}} u^2 dx + C_2 \cdot \varepsilon \leq d_0 \alpha \int\limits_{D_{\Sigma'}} |\nabla u|^2 dx + \frac{d_0 \cdot n \cdot M^2 \cdot mes_n D}{\alpha} + C_2 \cdot \varepsilon \tag{12}$$

Choosing $\alpha = \frac{\gamma}{d_0}$ from (12) we get

$$\int\limits_{D_{\Sigma'}} |\nabla u|^2 dx \leq C_3, \tag{13}$$

where C_3 depends on $M, d_0, \gamma, mes_n D, n.$ Without loss of generality we assume that $\varepsilon \leq 1.$ Hence we have

$$\int\limits_{D} |\nabla u|^2 dx \leq C_4,$$

where C_4 depends on C_3, E, D.

Further using (7) and (8) we obtain

$$\int\limits_{D_{\Sigma'}} e^{2Au}\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} dx = -\frac{1}{2A}\left[\int\limits_{D_{\Sigma'}} e^{2Au}\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j} u_{x_i} dx + \right.$$

$$\left. + \int\limits_{D_{\Sigma'}} e^{2Au}\sum_{j=1}^{n} b_i(x)\, u_{x_i} dx + \int\limits_{D_{\Sigma'}} e^{2Au} c(x)\, u dx\right], \tag{14}$$

for any $A \in [0,1]$. Using conditions (4), (6) and estimation (13) we get

$$\lim_{A\to 0}\int\limits_{D_{\Sigma'}} e^{2Au}\left[\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j} u_{x_i} + b_i(x)\, u_{x_i} dx + C(x)\, u\right] dx = 0.$$

Hence be Lebesgue theorem

$$\int\limits_{D_{\Sigma'}}\left[\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j} u_{x_i} + b_i(x)\, u_{x_i} + c(x)\, u\right] dx = 0. \tag{15}$$

Using (7), (8) and (15) we have

$$\int\limits_{D_{\Sigma'}} e^{2Au}\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} u_{x_j} dx \le C_5 \cdot \varepsilon.$$

From the last inequality by virtue of (4) it follows the assertion that $u \equiv const.$ So the following theorem is proved.

Theorem 1. *Let D be a bounded domain in R^n, $n \ge 2$, $E \subset \overline{D}$ be some compact. Relative to the coefficients conditions (4)-(6) be fulfilled. Then for removability of the compact E relative to the problem (1),(2) in the space $C^{0,\lambda}(D)$ it is sufficient that*

$$m_H^{n-2+\lambda}(E) = 0. \tag{16}$$

Consider the mixed boundary-value problem for non-divergent elliptic equation of the second order. Let Γ_1 and Γ_2 be such two sets that $\partial D\backslash E = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$. Then the boundary value problem

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u = 0 \qquad \text{in } D$$

$$u|_{\Gamma_1} = 0, \ \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_2} = 0 \tag{17}$$

is a mixed boundary value problem. The solution of boundary value problem (1), (17) we find from the classes $C^2(D) \cap C^0\left(\overline{D}\backslash E\right)$ and $\left\{W_2^1(D) \cap C^0\left(\overline{D}\backslash E\right);\ 0 \le u(x) \le k\right\}$.

Theorem 2. *Let D be a bounded domain in R^n, $n \geq 2$, $E \subset \overline{D}$ be some compact. Relative to the coefficients conditions (4)-(6)be fulfilled. Then for removability of the compact E relative to the problem (1), (17) it is sufficient that*

$$m_H^{n-2}(E) < \infty.$$

The theorem is proved by the same ideas as in theorem 1.

By the same methods that above we can consider one class of quasilinear elliptic equations of the second order. Consider in the domain $D \subset R^n$, $n \geq 2$ the following equation

$$\mathcal{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u + b(x, u, \bigtriangledown u) = 0. \qquad (18)$$

Here $a_{ij}(x)$ are measurable bounded functions satisfying the condition (4), $b_i(x)$, $c(x)$ satisfy condiiton (6), and

$$|b(x, u, \bigtriangledown u)| \leq g(u)\, |\bigtriangledown u|\,, \quad \int_0^a g(u)\, du < \infty, \ a < \infty. \qquad (19)$$

Consider the Neumann problem for equation (18):

$$Lu = 0, \ x \in D \backslash E, \ \left.\frac{\partial u}{\partial \nu}\right|_{\partial D \backslash E} = 0, \qquad (20)$$

The solution of problem (20) we seek in the class $W_2^1(D) \cap C^{0,\lambda}\left(\overline{D}\right)$, $|u| \leq k$.

Theorem 3. *Let D be a bounded domain in R^n, $n \geq 2$, $E \subset \overline{D}$ be some compact. Relative to the coefficients of equation (18) conditions (4), (6), (19) be fulfilled. Then for removability of the compact E relative to the problem (20) it is sufficient that*

$$m_H^{n-2+\lambda}(E) = 0.$$

Proof. Introduce the function

$$\vartheta(x) = \int_0^{u(x)} \exp\left(\frac{1}{\lambda_1}\int_0^t g(\tau)\, d\tau\right) dt.$$

Analogously [8] it is established that this function is a subsolution of the linear operator

$$\mathcal{L}_1 u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right).$$

Further, analogously to the proof of theorem 1 we get $\vartheta(x) = const$, i.e., $u(x) \equiv const$ that proves the theorem.

For mixed boundary value problem the following result is true.

Theorem 4. *At fulfilment of conditions of previous theorem, for removability of the compact E relative to the mixed boundary value problem (18), (17) it is sufficient that*

$$m_H^{n-2}(E) < \infty.$$

The theorem is proved with near ideas as in theorems 3 and 1.

Remark. The cited theorems are true for parabolic equations

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)\, u_{x_i} + c(x,t)\, u - u_t = 0 \qquad (21)$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x,t)\, u_{x_i} + c(x,t)\, u + b(t,x,u,\bigtriangledown u) - u_t = 0. \qquad (22)$$

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