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BASES OF EXPONENTS WITH DEGENERATE COEFFICIENTS

Abstract

In this paper the basicity theorem of some systems of exponents in the space $L_p(-\pi,\pi)$ is obtained.

Basicity in $L_2(-\pi,\pi)$ of classic system of exponents with degenerate coefficients $\left\{\frac{1}{\sqrt{2\pi}}t(x)e^{inx}\right\}$, $n=0,\pm1,...$, was considered earlier by V.F.Gaposhkin [1]. Concrete conditions, for carrying out of which relatively harmonic function, constructed with the help of this function, M.Riesz inequality takes place, are put on the function $\varphi(x)\equiv t^2(x)$.

The following system of exponents in $L_p\equiv L_p(-\pi,\pi)$

$$\left\{A^+(t)\,\omega^+(t)\,e^{int};\ A^-(t)\,\omega^-(t)\,e^{-ikt}\right\}_{n\geq0,k\geq1}, \tag{1}$$

which also covers the case, if $\varphi(x)$ has power form, is considered in the proposed paper. In our case the functions $\omega^{\pm}(t)$ have the following presentations

$$\omega^{\pm}(t)\equiv\prod_{i=1}^{l^{\pm}}\left\{\sin\left|\frac{t-\tau_i^{\pm}}{2}\right|\right\}^{\beta_i^{\pm}}, \tag{2}$$

where $\{\tau_i^{\pm}\}:-\pi\leq\tau_1^{\pm}<...<\tau_{l^{\pm}}^{\pm}<\pi$ are some sets, and

$$\{\tau_i^{\pm}\}\cap\{\tau_i^{-}\}=\{\varnothing\}, \tag{3}$$

$A^{\pm}(t)\equiv|A^{\pm}(t)|\,e^{i\alpha^{\pm}(t)}$ are complex-valued functions on $[-\pi,\pi]$. Note, that particular cases of system (1) are eigen-functions of discontinuous differential operators with degenerate coefficients. Earlier the basicity of the system (1) in L_p was studied by Bilalov B.T. (see, for example [2,3]) in the cases, if the functions $\omega^{\pm}(t)$ are absent.

We require satisfying the following condition.

1) $\alpha^{\pm}(t)$ are piecewise-Hōlder functions on $[-\pi,\pi]$, $\{s_i\}_1^r\subset[-\pi,\pi)$ is the set of discontinuity points of the function $\theta(t)\equiv\alpha^{\pm}(t)-\alpha^{-}(t)$. Moreover, $\{\tau_i^{\pm}\}\cap\{s_i\}_1^r=\{\varnothing\}$ and it takes place

$$0<\left\|A^{\pm};A^{-}\right\|_{\infty}^{\pm1}<+\infty,$$

where $\|\cdot\|_{\infty}$ is the norm in L_{∞}.

Denote by $\{h_i\}_1^r$ the jumps of the function $\theta(t)$ at the points $s_i: h_i=\theta(s_i+0)-\theta(s_i-0)$, $i=\overline{1,r}$.

Thus, the following theorem is true.

Theorem. *Let the following conditions be fulfilled*

$$-\frac{1}{p} < \beta_i^{\pm} < \frac{1}{q}, \ i = \overline{1, l^{\pm}}, \tag{4}$$

$$-\frac{2\pi}{q} < h_k < \frac{2\pi}{p}, \ k = \overline{1, r}. \tag{5}$$

Then system (1) forms the basis in L_p, $1 < p < +\infty$, *where* $q : \frac{1}{q} + \frac{1}{q} = 1$, *conjugate to* p *number.*

Before we prove this theorem, we introduce some classes of functions. Let $\nu^+(t)$ be some measurable, non-negative function on $(-\pi, \pi)$. H_δ^+, $\delta > 0$ is usual Hardy class of analytical functions in a unit circle. Introduce the following weight Hardy class H_{p,ν^+}^+:

$$H_{p,\nu^+}^+ \overset{def}{\equiv} \left\{ f \in H_1^+ : \int_{-\pi}^{\pi} \left| f^+ \left(e^{it}\right) \right|^p \nu^+(t)\, dt < +\infty \right\},$$

where $f^+(\tau)$ is nontangential boundary values of the function $f(z)$ at the point $\tau : |\tau| = 1$, inside the unit circle. Class H_{p,ν^-}^- is introduced analogously. Further, denote by $L_{p,\nu}$ the class of measurable on Lebesgue functions on $(-\pi, \pi)$ with the norm

$$\|f\|_{p,\nu} \equiv \left(\int_{-\pi}^{\pi} |f(t)|^p \nu(t)\, dt \right)^{1/p}.$$

So, let

$$\nu^{\pm} \equiv [\omega^{\pm}]^p, \tag{6}$$

$$G\left(e^{it}\right) \equiv \frac{\omega^-(t)\, A^-(t)}{\omega^+(t)\, A^+(t)}. \tag{7}$$

Consider the following conjugation problem in classes $H_{p,\nu^{\pm}}^{\pm}$:

$$\begin{cases} F^+(\tau) + G(\tau)\, F^-(\tau) = g(\arg \tau), & |\tau| = 1, \\ F^-(\infty) = 0. \end{cases} \tag{8}$$

Any pair of functions $\{F^+; F^-\} : F^{\pm} \in H_{p,\nu^{\pm}}^{\pm}$, boundary values $F^{\pm}(\tau)$ of which on the unit circle satisfy the equation (8) almost everywhere and F^- is equal to zero at infinity, is called the solution of conjugation problem (8) in class $H_{p,\nu^{\pm}}^{\pm}$.

As usually, the general solution of the problem (8) is presented in the form

$$F(z) = F_1(z) + F_0(z),$$

where $F_1(z)$ is a general solution of corresponding homogenous problem

$$F_1^+(\tau) + G(\tau)\, F_1^-(\tau) = 0, \ |\tau| = 1 \tag{9}$$

and $F_0(z)$ is any particular solution of nonhomogeneous problem (8).

First of all we'll find the general solution of homogeneous problem (9), which has the order $\leq m$ at infinity. Introduce the following analytical functions inside and outside the unit circle:

$$X_1^{\pm}(z)=\exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{\omega^-(t)}{\omega^{\pm}(t)}\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_1^{\pm}(z)=\exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|\frac{A^-(t)}{A^{\pm}(t)}\right|\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_1^{\pm}(z)=\exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

where $\theta(t)\equiv\alpha^-(t)-\alpha^+$. Let

$$Z_i(z)\equiv\begin{cases}X_i(z), & |z|<1,\\ [X_i^-(z)]^{-1}, & |z|>1,\ i=\overline{1,3}.\end{cases}$$

Formally we'll call the function $Z(z)\equiv\prod\limits_i Z_i(z)$ canonical solution of homogenous problem (9). So, the following lemma is true.

Lemma 1. *The general solution of homogeneous problem (9) in class $H^{\pm}_{p,\nu^{\pm}}$, $1<p<+\infty$ is presented in the form*

$$F_1(z)=Z(z)\cdot P_m(z),$$

if all conditions of theorem are fulfilled, where P_m is a polynomial of degree $\leq m$.

Applying Sokhotsky-Plemel formulae, we obtain that the equality

$$\frac{A^-(t)\,\omega^-(t)}{A^{\pm}(t)\,\omega^+(t)}=\frac{Z^+\left(e^{it}\right)}{Z^-\left(e^{it}\right)},\tag{10}$$

is fulfilled almost everywhere.

We present the function $\theta(t)$ in the form: $\theta(t)=\theta_0(t)+\theta_1(t)$, where $\theta_0(t)$ is continuous part, $\theta_1(t)$ is the jump function, which is determined by the formula: $\theta(t)\equiv\alpha^-(t)-\alpha^+(t)$,

$$\theta_1(-\pi)=0,\theta_1(t)=\sum_{-\pi<S_k<t}h_k+[\theta(t)+\theta(t-0)],$$

$$-\pi<t\leq\pi.$$

Let

$$h_0=h_0^{(1)}-h_0^0,$$

where

$$h_0^{(1)}=\theta_1(-\pi)-\theta_1(\pi),\ h_0^{(0)}=\theta_0(\pi)-\theta_0(-\pi).$$

Denote by $u(t) \equiv \prod\limits_k \left\{ \sin \left| \frac{t-s_k}{2} \right| \right\}^{-\frac{h_k}{2\pi}}$,

$$u_0(t) \equiv \left\{ \sin \left| \frac{t-\pi}{2} \right| \right\}^{\frac{h_0^0}{2\pi}} \cdot \exp \left\{ -\frac{1}{4} \int_{-\pi}^{\pi} \theta_0(s)\, ctg \frac{t-s}{2} ds \right\}.$$

Applying Sokhotsky-Plemel formulae, it is not difficult to obtain the following correlations:

$$\left| Z_1^- \left(e^{it} \right) \right| = \left[\frac{\omega^+(t)}{\omega^-(t)} \right]^{1/2},$$

$$0 < \left\| Z_2^- \left(e^{it} \right) \right\|_\infty < +\infty,$$

$$\left| Z_3^- \left(e^{it} \right) \right| = u_0(t) \cdot u(t) \cdot \left\{ \sin \left| \frac{t-\pi}{2} \right| \right\}^{-\frac{h_0}{2\pi}}.$$

As it follows from the results of the paper [4, p.79], the functions $u_0^{\pm}(t)$ are summable with any degree $p < +\infty$ on the segment $[-\pi, \pi]$. Further, we'll take into consideration (10) in (9). We have

$$\frac{F_1^+(\tau)}{Z^+(\tau)} = -\frac{F_1^-(\tau)}{Z^-(\tau)}, \quad |\tau| = 1.$$

We introduce the new piecewise-analytical function $\Phi(z)$:

$$\Phi(z) = \begin{cases} \dfrac{F_1^+(z)}{Z^+(z)}, & |z| < 1, \\ -\dfrac{F_1^-(z)}{Z^-(z)}, & |z| > 1. \end{cases}$$

As $Z(z)$ has no zeros and poles for $|z| \neq 1$, then the functions $\Phi(z)$ and $F_1(z)$ have the same orders at the infinity. We'll investigate belonging of the function $\Phi(z)$ to the class $H_1^{\pm}$. According to definition $F_1^+ \in H_1^+$. Moreover, $Z(z) \in H_\delta^{\pm}$ for sufficiently small $\delta > 0$. Consequently, $\Phi^+ \in H_\mu^+$ for some $\mu > 0$. According to definition $F_1 \in H_1^{\pm}$. On the other hand $F_1^- \left(e^{it} \right) \cdot \omega^-(t) \in L_p$. It remains to investigate belonging of the function $\left[Z^- \left(e^{it} \right) \cdot \omega^-(t) \right]$ to the class L_q, $\frac{1}{p} + \frac{1}{q} = 1$. And it follows directly from the lemma conditions. As a result we have $\Phi(\tau) \in L_1(\Gamma)$, $\Gamma : |\tau| = 1$, and thus $\Phi(z) \in H_1^{\pm}$. Consequently, $\Phi(z)$ is a polynomial of degree $\leq m$, i.e. $\Phi(z) \equiv P_m(z)$, and as a result

$$F_1(z) \equiv Z(z) \cdot P_m(z). \tag{11}$$

Now we'll show that $F_1(z) \in H_{p,\nu^{\pm}}^{\pm}$. Again from the conditions of theorem it follows that $F_1^{\pm} \left(e^{it} \right) \cdot \omega^{\pm}(t) \in L_p$ and $Z^{\pm} \left(e^{it} \right) \in L_1$. From (11) and from Smirnov theorem we have $F_1(z) \in H_1^{\pm}$. Thus, according to definition $F_1 \in H_{p,\nu}^{\pm}$. Lemma is proved.

And now let's prove the theorem. Let's research the solution of non-homogeneous conjugation problem (8) in classes $H^{\pm}_{p,\nu}$. Consider the function

$$F_0(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{g(\sigma)}{Z^+(e^{i\sigma})} \cdot \frac{d\sigma}{1 - z \cdot e^{-i\sigma}}, \tag{12}$$

where $Z(z)$ is a canonical solution of homogenous problem. Sokhotsky-Plemel formulae direct application gives us that the boundary values of the function $F_0(z)$ satisfy the equation (8) almost everywhere and $F_0(\infty) = 0$, i.e.

$$F_0^+\left(e^{it}\right) + \frac{A^-(t) \cdot \omega^-(t)}{A^+(t) \cdot \omega^+(t)} \cdot F_0^-\left(e^{it}\right) = g(t), \quad t \in (-\pi, \pi),$$

where $g \in L_{p,\nu^+}$ is an arbitrary function. Denote by $f(t) \equiv g(t) \cdot \omega^+(t)$. It is clear, that $f \in L_p$. Let $Z_0\left(e^{it}\right) \equiv Z^+\left(e^{it}\right) \cdot \omega^+(t)$.

Applying Sokhotsky-Plemel formulae and transforming, we obtain:

$$F_0^+\left(e^{it}\right) \cdot \omega^+(t) = f(t) + \frac{Z_0\left(e^{it}\right)}{2\pi} \int_{-\pi}^{\pi} \frac{g(\sigma)}{Z_0(e^{i\sigma})} \cdot \frac{d\sigma}{1 - e^{i(t-\sigma)}}.$$

From this presentation according to theorem 8.4 [4, p. 141] it follows that $F_0^+\left(e^{it}\right) \cdot \omega^+(t) \in L_p$, i.e. $F_0^+ \in L_{p,\nu^+}$. As $g(\sigma) \cdot \left[Z^+\left(e^{i\sigma}\right)\right]^{-1} \in L_1$, then it is clear that

$$\int_{-\pi}^{\pi} \frac{g(\sigma)}{Z^+(e^{i\sigma})} \frac{d\sigma}{1 - z e^{i\sigma}} \in H_1^{\pm}.$$

Consequently, $F_0 \in H^+_\mu$ for some $\mu > 0$. From $[\omega^+(t)]^{-1} \in L_q$ we obtain that $F_0^+\left(e^{it}\right) \in L_1$ and as a result from Smirnov theorem $F_0^+(z) \in H_1$. Thus, $F_0^+(z) \in H^+_{p,\nu^+}$. Analogously we can prove that $F_0^-(z) \in H^-_{p,\nu^-}$. As a result the constructed function is the solution. From $F(\infty) = 0$ it follows that homogeneous problem has only trivial solution, consequently, non-homogeneous problem has a unique solution (12). Thus, we obtain the following conclusion.

If all conditions of theorem are fulfilled, then conjugation problem (8) in classes $H^{\pm}_{p,\nu^\pm}$ has a unique solution in the form (12) for $\forall g \in L_{p,\nu^+}$. Further we'll show that any function f from L_p has decomposition on system (1) in L_p. First of all we'll introduce some subspaces L_p. Let $H^+_{p;m}$ H^-_p be usual Hardy classes of analytical functions inside and outside the unit circle, correspondingly, where m is the order of the main part of decomposition of the function from H^-_p into Laurent series at infinity. Denote by L^+_p and L^-_p the restrictions of the functions from H^+_p and H^-_p correspondingly of the unit circle. It is not difficult to note that L^+_p and L^-_p are the subspaces of the space $L_p(-\pi, \pi)$. Consider the following weight subspaces:

$$L^{+def}_{p,\nu} \equiv \left\{ f \in L_1^+ : \|f\|_{p,\nu} < +\infty \right\},$$

$$_mL^{-def}_{p,\nu} \equiv \left\{ f \in_m L_1^- : \|f\|_{p,\nu} < +\infty \right\},$$

where $\|f\|_{p,\nu}^{p} \overset{def}{\equiv} \int\limits_{-\pi}^{\pi} |f(t)|^{p} \nu(t)\, dt$, $\nu(t)$ is a measurable, almost everywhere non-negative function. Using one result of Babenko K.I. [5] we can prove the following lemma.

Lemma 2. *Let* $\mu^{\pm}(x) = \prod\limits_{i=0}^{l^{\pm}} \left|x - x_i^{\pm}\right|^{\alpha_i^{\pm}}$, *where* $-\pi \le x_0^{\pm} < x_1^{\pm} < ... < x_{l^{\pm}}^{\pm} < \pi$, $-1 < \alpha_i^{\pm} < p-1$, $\forall i = \overline{0, l^{\pm}}$. *Then the system* $\{e^{int}\}_{n\ge 0}$ $\left(\{e^{-int}\}_{n\ge m}\right)$ *forms the basis in the spaces* $L_{p,\mu}^{+}$ $\left({}_{m}L_{p,\mu^{-}}^{-}\right)$, $1 < p < +\infty$.

Passing to the proof of the theorem we note that $F_0^{+}\left(e^{it}\right) \in L_{p,\nu^{+}}^{+}$ and $F_0^{-} \in {}_{1}L_{p,\nu}^{-}$. From the conditions (4) it follows that the numbers $\alpha_i^{\pm} = p \cdot \beta$, $i = \overline{0, l^{\pm}}$, satisfy the conditions of lemma 2. Then by this lemma the systems $\{e^{int}\}_{n\ge 0}$, $\{e^{-int}\}_{n\ge m}$ from the bases of subspaces $L_{p,\nu^{+}}^{+}$ and $L_{p,\nu^{-}}^{-}$ correspondingly. Decomposing the functions $F_0^{+}\left(e^{it}\right)$ and $F_0^{-}\left(e^{it}\right)$ on this systems we obtain that any function from L_p can be decomposed on system (1). As following step we'll prove the minimality of system (1) in L_p. For this we take $g(t) \equiv e^{int}$, $n \ge 0$ as the function $g(t)$ in equation (8). As it was already shown, a unique solution of this problem in the class $H_{p,\nu^{\pm}}^{\pm}$ has the presentation (12).

We decompose the function (12) in series on the powers of z in zero and point at infinity. So, let

$$Z(z) = \sum_{n=0}^{\infty} a_n^{+} \cdot z^{n}, \quad |z| < 1,$$

$$Z(z) = \sum_{n=0}^{\infty} a_n^{-} \cdot z^{-n}, \quad |z| > 1.$$

We have

$$I(z) \equiv \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \frac{e^{-in\sigma}}{2\pi Z^{+}\left(e^{i\sigma}\right)} \cdot g(\sigma)\, d\sigma \cdot Z^{n}, \quad \text{for} \quad |z| < 1,$$

and

$$I(z) \equiv \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \frac{e^{-i(n-1)\sigma}}{2\pi Z^{+}\left(e^{i\sigma}\right)} \cdot g(\sigma)\, d\sigma \cdot Z^{-n}, \quad \text{for} \quad |z| > 1,$$

where

$$I(z) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(\sigma)}{Z^{+}\left(e^{i\sigma}\right)} \frac{d\sigma}{1 - z \cdot e^{-i\sigma}}.$$

Multiplying the corresponding decompositions, grouping by the powers of z and introducing the notation

$$\bar{h}_n^{+}(t) = \frac{\varphi_n^{+}(t)}{Z^{+}\left(e^{it}\right)}, \quad n \ge 0,$$

$$\bar{h}_n^{-}(t) = \frac{\varphi_n^{-}(t)}{Z^{+}\left(e^{it}\right)}, \quad n \ge 1,$$

where $\varphi_n^{\pm}$ are corresponding sums in the integrand, we have:

$$F_0(z) = \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \bar{h}_n^+(t) \cdot g(t)\, dt \cdot z^n, \text{ for } |z| < 1,$$

and

$$F_0(z) = \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \bar{h}_k^-(t) \cdot g(t)\, dt \cdot z^{-n}, \text{ for } |z| > 1,$$

where $\bar{(\cdot)}$ is complex conjugation.

From the conditions (4), (5) and the presentations for $Z^+(e^{it})$ it follows that $Z^+(e^{int}) \in L_q$. From the other side it is easy to note that the function

$$F_0(z) \equiv \begin{cases} z^n, & |z| < 1, \\ 0, & |z| > 1, \end{cases}$$

is also the solution of the problem (8) in the classes $H^{\pm}_{p,\nu^{\pm}}$. From comparisons of the corresponding coefficients we obtain:

$$\int_{-\pi}^{\pi} \bar{h}_k^+(t) \cdot \omega^+(t)\, e^{int} dt = \delta_{nk}, \quad \forall_{n,k \geq 0},$$

$$\int_{-\pi}^{\pi} \bar{h}_k^+(t) \cdot \omega^-(t)\, e^{int} dt = 0, \ \forall k \geq 0, \ \forall n \geq 1.$$

Analogous analysis gives us:

$$\int_{-\pi}^{\pi} \bar{h}_k^+(t) \cdot \omega^+(t)\, e^{int} dt = 0, \ \forall k \geq 1, \ \forall n \geq 0,$$

and

$$\int_{-\pi}^{\pi} \bar{h}_k^- \cdot \omega^-(t)\, e^{-int}\, dt \geq \delta_{nk}, \ \forall_{n,k \geq 1},$$

where δ_{nk} is Kronecker symbol. Then the system $\{h_n^+(t); h_k^-(t)\}_{n \geq 0, k \geq 1}$ is biorthogonal to the system (1) and, consequently, (1) is minimal in L_p. As a result we obtain the basicity of the system (1) in L_p.

The theorem is proved.

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