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TO THE THEORY OF WAVE PROPAGATION IN ANISOTROPIC STRUCTURES

Abstract

In the paper the general, three-dimensional equations of transversally-isotropic bodies and the characters of distribution of different waves in them, are investigated. Continuing the method investigated by the author for isotropic elastic bodies [2] it was succeeded to decompose equations relative to potentials. The solution of these equations for simple case was obtained.

In the paper the general equations of dynamics of transversally-isotropic bodies are investigated with the position of exact theory. At this the ability of applying the method, investigated in (2) ant its consequences for the given case is studied.

Consider a linearly anistrophic body having the following properties: through the each point passes a surface of elastic symmetry in which all directions are elastico-equivalent (isotropic surface). In other words at each point there is one main direction and infinite set of main directions in surface, normal to the first. The body with such properties is called transversally-isotropic (1).

Let's direct the axis Z normally to the surface of isotropy and the axis X and Y arbitrarily in the same surface. Then the equations of Hook's generalized principle will take the form:

$$\varepsilon_x = \frac{1}{E_1}\left(\sigma_x - \nu_1\sigma_y - \nu_2\sigma_z\right)$$

$$\varepsilon_y = \frac{1}{E_1}\left(\sigma_y - \nu_1\sigma_x - \nu_2\sigma_z\right)$$

$$\varepsilon_z = \frac{1}{E_1}\left(-\nu_2\sigma_x - \nu_2\sigma_y + \frac{E_1}{E_2}\sigma_z\right) \tag{1}$$

$$\gamma_{yz} = \frac{1}{G_2}\tau_{yz}$$

$$\gamma_{xz} = \frac{1}{G_2}\tau_{xz}$$

$$\gamma_{xy} = \frac{1}{G_1}\tau_{xy}$$

where the "technical constants" are introduced.

E_1, E_2 are Young modulus for the expansion-compression in direction normal to the surface and in isotropy surface. ν_1, ν_2 are Poisson coefficients characterizing the saving in isotropy surface and in perpendicular direction at extension in isotropy surface, $G_1 = E_1/2\left(1+\nu_1\right)$, G_2 are shear modulus for the isotropy surface and for the surface perpendicular to it.

By formula (1) we can easily pass to the inverse dependences:

$$\sigma_x = \lambda_1\varepsilon_x + \mu_1\varepsilon_y + \mu_2\varepsilon_z; \quad \tau_{yz} = G_2\gamma_{yz}$$

$$\sigma_y = \mu_1\varepsilon_x + \lambda_1\varepsilon_y + \mu_2\varepsilon_z; \quad \tau_{xz} = G_2\gamma_{xz} \tag{2}$$
$$\sigma_z = \mu_2\varepsilon_x + \mu_2\varepsilon_y + \lambda_2\varepsilon_z; \quad \tau_{xy} = G_1\gamma_{xy}$$

where the notation

$$\lambda_1 = \frac{E_1\left(E_1 - \nu_2^2 E_2\right)}{\left[E_1\left(1-\nu_1\right) - 2\nu_2^2 E_2\right]\left(1+\nu_1\right)}$$
$$\mu_1 = \frac{E_1\left(E_1\nu_1 + E_2\nu_2^2\right)}{\left[E_1\left(1-\nu_1\right) - 2\nu_2^2 E_2\right]\left(1+\nu_1\right)} \tag{3}$$
$$\lambda_2 = \frac{2E_2^2\nu_2^2}{E_1\left(1-\nu_1\right) - 2\nu_2^2 E_2} + E_2 = \frac{E_2E_1\left(1-\nu_1\right)}{E_1\left(1-\nu_1\right) - 2\nu_2^2 E_2}$$
$$\mu_2 = \frac{\nu_2 E_2}{E_1}\left(\lambda_1 + \mu_1\right) = \frac{\nu_2 E_2 E_1}{E_1\left(1-\nu_1\right) - 2\nu_2^2 E_2}$$

are accepted.

The motion equations in permutations subject to (2) and (3) have the form:

$$\lambda_1\frac{\partial^2 u}{\partial x^2} + \left(\mu_1 + G_1\right)\frac{\partial^2 \upsilon}{\partial x\partial y} + \left(G_2 + \mu_2\right)\frac{\partial^2 w}{\partial x\partial z} + G_1\frac{\partial^2 u}{\partial y^2} + G_2\frac{\partial^2 u}{\partial z^2} = \rho\frac{\partial^2 u}{\partial t^2},$$

$$\left(\mu_1 + G_1\right)\frac{\partial^2 u}{\partial x\partial y} + \lambda_1\frac{\partial^2 \upsilon}{\partial y^2} + \left(G_2 + \mu_2\right)\frac{\partial^2 w}{\partial y\partial z} + G_1\frac{\partial^2 \upsilon}{\partial x^2} + G_2\frac{\partial^2 \upsilon}{\partial z^2} = \rho\frac{\partial^2 \upsilon}{\partial t^2}, \tag{4}$$
$$\left(G_2 + \mu_2\right)\frac{\partial^2 u}{\partial x\partial z} + \left(G_2 + \mu_2\right)\frac{\partial^2 \upsilon}{\partial y\partial z} + \lambda_2\frac{\partial^2 w}{\partial z^2} + G_2\frac{\partial^2 w}{\partial x^2} + G_2\frac{\partial^2 w}{\partial y^2} = \rho\frac{\partial^2 w}{\partial t^2}.$$

At first apply the Laplace integral transformation by t, and then according to [2] we make the substitution:

$$\bar{u} = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y} + \frac{\partial^2\psi_2}{\partial z\partial x}$$

$$\bar{\upsilon} = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi_1}{\partial x} + \frac{\partial^2\psi_2}{\partial z\partial y} \tag{5}$$
$$\bar{w} = \frac{\partial\varphi}{\partial z} - \frac{\partial^2\psi_2}{\partial x^2} - \frac{\partial^2\psi_2}{\partial y^2}$$

where the little line under the sign indicates the Laplace transformations of functions.

Then the system (4) takes the most simple form:

$$\lambda_1\frac{\partial}{\partial x}\left(H_1\varphi\right) + G_1\frac{\partial}{\partial y}\left[H_2\psi_1\right] + \left(\lambda_1 - \mu_2 - G_2\right)\frac{\partial^2}{\partial x\partial z}[H_3\psi_2] = 0,$$

$$\lambda_1\frac{\partial}{\partial y}\left(H_1\varphi\right) + G_1\frac{\partial}{\partial x}\left[H_2\psi_1\right] + \left(\lambda_1 - \mu_2 - G_2\right)\frac{\partial^2}{\partial y\partial z}[H_3\psi_2] = 0, \tag{6}$$
$$\frac{\partial}{\partial z}\left(2G_2 + \mu_2\right)\left[H_4\varphi\right] - G_2\Delta[H_5\psi_2] = 0$$

where the following operators

$$H_1=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{2G_2+\mu_2}{\lambda_1}\frac{\partial^2}{\partial z^2}-\frac{\rho}{2\lambda_1}p^2,$$

$$H_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{G_2}{G_1}\frac{\partial^2}{\partial z^2}-\frac{\rho}{G_1}p^2,$$

$$H_3=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{G_2}{\lambda_1-G_2-\mu_2}\frac{\partial^2}{\partial z^2}-\frac{\rho}{\lambda_1-G_2-\mu}p^2, \qquad (7)$$

$$H_4=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\lambda_2}{2G_2+\mu_2}\frac{\partial^2}{\partial z^2}-\frac{\rho}{2G_2+\mu_2}p^2,$$

$$H_5=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\lambda_2-\mu_2-G_2}{G_2}\frac{\partial^2}{\partial z^2}-\frac{\rho}{G_2}p^2,$$

are denoted by $H_i\left(i=\overline{1,5}\right)$.

$\Delta=\frac{\partial^2}{x^2}+\frac{\partial^2}{\partial y^2}$ is Laplace operator.

From the first two equations of system (6) we obtain:

$$\lambda_1H_1\varphi+(\lambda_1-\mu_2-b_1)\frac{\partial}{\partial z}(H_3\psi_2)=0$$

$$H_2\psi_1=0$$

which indicates on ability of propagation of rotating waves described by the potentials ψ_1, regardless of other waves: longitudinal φ and transversal ψ_2 if the rotation happens in isotropy surface.

Besides, from this system the speeds of different waves in different directions become known.

Unlike the isotropic bodies in anisotropic bodies the propagation of waves of the given type (φ or ψ_2) is impossible, regardless of other ψ_2 and φ in the space. Existence of boundaries makes more difficult the motion picture.

We'll investigate complex of system (6) concretely in the example of simple case. Exactly, we'll consider the rectangular prism of finite length from described medium which is subjected to the action of opposite axial forces, uniformly distributed in end cross-sections.

The lateral surfaces are subjected to the conditions of moving contact. All these conditions are mathematically expressed by the following form:

$$\left.\begin{array}{l}\bar{u}=\bar{v}=0\\ \bar{\sigma}_{zz}=\sigma_0\bar{f}(p)\end{array}\right\}\quad \text{at } z=\pm c \qquad (8)$$

$$\left.\begin{array}{l}\bar{u}=0\\ \bar{\sigma}_{xy}=0\\ \bar{\sigma}_{xz}=0\end{array}\right\}\quad \text{at } x=\pm a \qquad (9)$$

$$\left.\begin{array}{l}\bar{v}=0\\ \bar{\sigma}_{xy}=0\\ \bar{\sigma}_{yz}=0\end{array}\right\}\quad \text{at y}=\pm\, b \qquad (10)$$

Obviously, conditions (9) and (10) satisfy the functions of the form:

$$\varphi = \sum_{k,m} A_{km} \cos\alpha_k x \cos\beta_m y Z_0(z)$$

$$\psi_1 = \sum_{k,m} B_{km} \sin\alpha_k x \sin\beta_m y Z_1(z) \qquad (11)$$

$$\psi_2 = \sum_{k,m} C_{km} \cos\alpha_k x \cos\beta_m y Z_2(z)$$

$$\alpha_k = \frac{k\pi}{a}; \quad \beta_m = \frac{m\pi}{b},$$

At first from the equation $H_2\psi_1 = 0$ we'll get $\psi_1 \equiv 0$ at once. Since $B_{km} = 0\,(k,m = 0,\infty)$, then for the determination $Z_0(z)$ and $Z_2(z)$ we have the following equations:

$$\lambda_1 \left[\frac{2G_2+\mu_2}{\lambda_1} Z_0''(z) - \left(\frac{\rho p^2}{\lambda_1} + \gamma_{km}^2\right) Z_0(z)\right] + \frac{\partial}{\partial z}(\lambda_1 - G_2 - \mu_2) \times$$

$$\times \left[\frac{G}{\lambda_1 - G_2 - \mu_2} Z_2''(z) - \left(\frac{\rho p^2}{\lambda_1 - G_2 - \mu_2} + \gamma_{km}^2\right) Z_2(z)\right] = 0$$

$$(2G_2+\mu_2)\frac{\partial}{\partial z}\left[\frac{\lambda_2}{2G_2+\mu_2} Z_0''(z) - \left(\frac{\rho p^2}{2G_2+\mu_2} + \gamma_{km}^2\right) Z_0(z)\right] + G_2\gamma_{km}^2 \times$$

$$\times \left[\frac{\lambda_2 - \mu_2 - G_2}{G_2} Z_0'' - \left(\frac{\rho p^2}{G_2} + \gamma_{km}^2\right) Z_2(z)\right] = 0, \quad \gamma_{km}^2 = \alpha_k^2 + \beta_m^2 \qquad (12)$$

The corresponding characteristic equation is the following:

$$\begin{vmatrix} \lambda_1\left[\frac{2G_2+\mu_2}{\lambda_1}k^2 - \left(\frac{\rho p^2}{\lambda_1} + \gamma_{km}^2\right)\right] & k(\lambda_1 - G_2 - \mu_2)\left[\frac{G_2}{\lambda_1 - G_2 - \mu_2}k^2 - \left(\frac{\rho p^2}{\lambda_1 - G_2 - \mu_2}\right)\right] \\ (2G_2+\mu_2)\,k\left[\frac{\lambda_2}{2G_2+\mu_2}k^2 - \left(\frac{\rho p^2}{2G_2+\mu_2} + \gamma_{km}^2\right)\right] & G_2\gamma_{km}^2\left[\frac{\lambda_2 - \mu_2 - G_2}{G_2}k^2 - \left(\frac{\rho p^2}{G_2} + \gamma_{km}^2\right)\right] \end{vmatrix} = 0 \quad (13)$$

But as it is known from the conditions

$$\sigma_{zz}|_{zz\pm c} = \sigma_0 \bar{f}(p) \qquad \text{where } \sigma_0 = const$$

If we put here the expression of form (11) then only the coefficients A_{00} and C_{00} will differ from zero, but then $\gamma_{km} = 0$ and the characteristic equation has the roots:

$$k_{1;2} = \pm\sqrt{\frac{\rho}{G_2}}p \qquad (14)$$

$$k_{3;4} = \pm\sqrt{\frac{\rho}{\lambda_2}}p$$

$$k_{5;6} = 0$$

then

$$Z_0(z) = A_{11}ch\sqrt{\frac{\rho}{\lambda_2}}pz + A_{12}ch\sqrt{\frac{\rho}{G_2}}pz$$

$$Z_2(z) = A_{21} sh\sqrt{\frac{\rho}{\lambda_2}}pz + A_{22} sh\sqrt{\frac{\rho}{G_2}}pz \tag{15}$$

Substituting (15) in equation (12) we'll obtain:

$$A_{12} = 0$$

$$A_{21} = A_{11}\frac{2b_2+\mu_2}{\rho}\cdot\frac{c_{12}}{p}\cdot\frac{c_{12}^{-2}-c_{21}^{-2}}{c_{12}^{-2}-c_{11}^{-2}}$$

$$c_{11}^2 = \frac{2G_2+\mu_2}{\rho};\quad c_{12}^2 = \frac{\lambda_2}{\rho};\quad c_{21}^2 = \frac{G_2}{\rho};$$

or

$$\varphi = A_{11} ch\frac{p}{c_{12}}z$$

$$\psi_2 = A_{21} sh\frac{p}{c_{12}}z + A_{22} sh\frac{p}{c_{21}}z.$$

Since ψ_2 is independent of transversal coordinates x, y then this function judging by formulae (5) doesn't beyond any permutations. The process is described only by the potential φ:

$$\varphi = \frac{\sigma_0 f(p)}{\rho}\cdot\frac{1}{p^2}\cdot\frac{ch\frac{p}{c_{12}}z}{ch\frac{p}{c_{12}}c} \tag{16}$$

which is determined from the conditions on prism end:

$$\bar{\sigma}_{zz}\big|_{z\pm c} = \sigma_0\bar{f}(p)$$

Solution (16) makes only $\bar{W}$:

$$W = \frac{\sigma_0 f(p)}{\lambda_2}\cdot\frac{c_{12}}{p}\cdot\frac{sh\frac{p}{c_{12}}z}{ch\frac{p}{c_{12}}c},$$

Turn to inverse transformations:

$$W=\frac{c_{12}\sigma_0}{\lambda_2}\left\{f(t) * \sum_k (-1)^k\left[\frac{\cos\left(\frac{1}{2}+k\right)\frac{\pi}{c}(z-c_{12}t)-\cos\left(\frac{1}{2}+k\right)\frac{\pi}{c}(z+c_{12}t)}{2k+1}\right]\right\} \tag{17}$$

Solution (17) represents the waves beginning from the both ends and passing in opposite directions with the speed $c_{12} = \sqrt{\lambda_2/\rho}$ towards to each other and being reflected from them.

References

[1]. Lekhnitskii S.G. *Elasticity theory of anisotropic body.* Gostekhizdat, Moscow, 1950. (Russian)

[2]. Rassoulova N.B. *Wave propagation in prismatic beam subjected to the action of axial forces.* Mekh. Tverdogo Tela, 1997, No6, pp.176-179. (Russian)

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