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CHARACTERISTIC OF SPECTRAL DATA OF DIRAC OPERATORS

Abstract

In the paper we found necessary and sufficient conditions to which must satisfy a collection of some quantities in order that it be spectral data of two self-adjoint boundary value problems generated on the segment by Dirac equation and non-separated boundary conditions.

Consider the boundary value problem $D(\omega,\beta,\gamma)$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}+\begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}=$$

$$=\lambda\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\quad x\in[0,\pi], \tag{1}$$

$$\begin{pmatrix} \beta & 1 \\ -\bar{\omega} & 0 \end{pmatrix}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}+\begin{pmatrix} \omega & 0 \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} y_1(\pi) \\ y_2(\pi) \end{pmatrix}=0,$$

where $p(x)$ and $q(x)$ are real functions from $L_2[0,\pi]$, ω is a complex number, and β,γ are real numbers.

In the paper we consider the case when $|\omega|^2\neq\gamma^2+1$.

The characteristic of spectra of two boundary value problems of the form $D(\omega,\beta_1,\gamma)$, $D(\omega,\beta_2,\gamma)$ in the case $\omega^2=\gamma^2+1$ $(\operatorname{Im}\omega=0)$ was obtained in [1]. The inverse problem on reconstruction of the boundary value problems $D(0,\beta_1,\gamma)$ and $D(0,\beta_2,\gamma)$ was solved in [2]. The inverse periodic problem was studied in [3-4] by different methods.

Note that the main theorem on the inverse problem in the considered case without proof was announced in author's paper [5].

By $\begin{pmatrix} c_1(\lambda,x) \\ c_2(\lambda,x) \end{pmatrix}$ and $\begin{pmatrix} s_1(\lambda,x) \\ s_2(\lambda,x) \end{pmatrix}$ we denote the solution of equation (1) satisfying the conditions $\begin{pmatrix} c_1(\lambda,0) \\ c_2(\lambda,0) \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} s_1(\lambda,0) \\ s_2(\lambda,0) \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Using the identity

$$c_1(\lambda,x)\,s_2(\lambda,x)-c_2(\lambda,x)\,s_1(\lambda,x)\equiv 1, \tag{2}$$

we easily get that the characteristic fucntion of the problem $D(\omega,\beta,\gamma)$ is of the form

$$d(\lambda)=V_+(\lambda)+\beta\left[s_2(\lambda,\pi)+\gamma s_1(\lambda,\pi)\right]+2\operatorname{Re}\omega, \tag{3}$$

where

$$V_+(\lambda)=|\omega|^2 s_1(\lambda,\pi)-c_2(\lambda,\pi)-\gamma c_1(\lambda,\pi). \tag{4}$$

Consider the function

$$V_-(\lambda)=-|\omega|^2 s_1(\lambda,\pi)-c_2(\lambda,\pi)-\gamma c_1(\lambda,\pi). \tag{5}$$

We can easily be convinced that

$$V_-(\nu_k) = signV_-(\nu_k)\sqrt{V_+^2(\nu_k) - 4|\omega|^2}, \tag{6}$$

where ν_k $(k = 0, \pm1, \pm2, ...)$ are the zeros of the function $s_2(\lambda, \pi) + \gamma s_1(\lambda, \pi)$, i.e. eigen values of the boundary value problem generated by equation (1) and boundary conditions

$$y_1(0) = y_2(\pi) + \gamma y_1(\pi) = 0. \tag{7}$$

By (2) and (5) we have

$$V_-(\nu_k) = \frac{1}{s_1(\nu_k, \pi)} - |\omega|^2 s_1(\nu_k, \pi) = \frac{1 - |\omega|^2 s_1^2(\nu_k, \pi)}{s_1(\nu_k, \pi)}. \tag{8}$$

Spectral data of two boundary value problems $D(\omega, \beta_1, \gamma)$, $D(\omega, \beta_2, \gamma)$ are the totality of their spectra, sequences of signs $\{\sigma_k\}$ $(\sigma_k = sign(1 - |\omega s_1(\lambda_k, \pi)|))$ and the number ω.

Theorem. *In order the number ω and sequences $\left\{c_{1,k}^{\pm}\right\}$ and $\left\{c_{2,k}^{\pm}\right\}$ and $\{\sigma_k\}$ be spectral data of boundary value problems of the form $D(\omega, \beta_1, \gamma)$, $D(\omega, \beta_2, \gamma)$ $\left(\beta_1 < \beta_2,\ |\omega|^2 < \gamma^2 + 1\right)$ it is necessary and sufficient that the following conditions be fulfilled:*

$$1)\quad c_{j,k}^{\pm} = 2k + r_j^{\pm} + \delta_{j,k}^{\pm}, \tag{9}$$

where $r_j^{\pm} = \frac{2}{\pi} arctg \frac{a_j \pm \sqrt{a_j^1 + b_j^2 - 4r^2}}{2r - b_j}$, $a_j = \beta_j\gamma + |\omega|^2 + 1$, $b_j = \beta_j - \gamma$, $r = \operatorname{Re}\omega$ *are real numbers,* $\sum\limits_{k=-\infty}^{\infty}\left(\delta_{j,k}^{\pm}\right)^2 < \infty$

2) the numbers $c_{1,k}^{\pm}, c_{2,k}^{\pm}$ $(k = 0, \pm1, \pm2, ...)$ for $\operatorname{Im}\omega \neq 0$ alternate:

$$... < c_{1,k}^- < c_{2,k}^- < c_{1,k}^+ < c_{2,k}^+ < c_{1,k+1}^- < c_{2,k+1}^- < ...,$$

and for $\operatorname{Im}\omega = 0$ satisfy the inequalities

$$... \le c_{1,k}^- \le c_{2,k}^- \le c_{1,k}^+ \le c_{2,k}^+ \le c_{1,k+1}^- \le c_{2,k+1}^- \le ...,$$

moreover, if two sequential terms of the sequence $\left\{c_{1,k}^{\pm}\right\}$ $\left(\left\{c_{2,k}^{\pm}\right\}\right)$ are equal, then the term of the sequence $\left\{c_{2,k}^{\pm}\right\}$ $\left(\left\{c_{1,k}^{\pm}\right\}\right)$ coinciding with these two terms differ from other terms of the sequence $\left\{c_{1,k}^{\pm}\right\}$ $\left(\left\{c_{2,k}^{\pm}\right\}\right)$

3)

$$|c_k| \ge 2|\omega|, \tag{10}$$

where $c_k = d_j(\nu_k) - 2r$,

$$d_j(z) = (b_j + 2r)\prod_{k=-\infty}^{\infty}\frac{\left(c_{j,k}^- - z\right)\left(c_{j,k}^+ - z\right)}{\left(2k + r_j^-\right)\left(2k + r_j^+\right)}, \tag{11}$$

ν_k are the zeros of the function $d_1(z) - d_2(z)$; $j = 1, 2$;

4) $\sigma_k = 0$ if $|c_k| = 2|\omega|$, and $\sigma_k = \pm 1$ otherwise, and there exists such a natural number N that $\sigma_k = 1$ for $|k| > N$.

Proof. Necessity. Let $\left\{c^{\pm}_{1,k}\right\}$, $\left\{c^{\pm}_{2,k}\right\}$ and $\{\sigma_k\}$, ω be spectral data of boundary value problems $D(\omega, \beta_1, \gamma)$, $D(\omega, \beta_2, \gamma)$. The necessity of the first and second conditions was established in [6]. By relation (2), (3) and (4) we have

$$c_k = d_j(\nu_k) - 2r = V_+(\nu_k) = |\omega|^2 s_1(\nu_k, \pi) - c_2(\nu_k, \pi) - \gamma c_1(\nu_k, \pi) =$$

$$= |\omega|^2 s_1(\nu_k, \pi) + \frac{1}{s_1(\nu_k, \pi)} .$$

Hence, the validity of inequality (10) is easily obtained.

If $|c_k| = 2|\omega|$, then $V_+^2(\nu_k) = 4|\omega|^2$. Then by (6) $V_-(\nu_k) = 0$ holds.

Taking formula (8) into account we get

$$\sigma_k = sign\left(1 - |\omega s_1(\nu_k, \pi)|\right) = (-1)^{k+1} sign V_-(\nu_k) = 0.$$

Let $|c_k| \neq 2|\omega|$. Then, it is clear that σ_k attains -1 or 1. Using the representation of the fucntion $s_1(\lambda, \pi)$ (see [3]) and taking into account the asymptotic formula $\nu_k = k + \frac{1}{\pi} arcctg\ \gamma + m_k$, $\sum\limits_{k=-\infty}^{\infty} m_k^2 < \infty$ and the inequality $|\omega|^2 < \gamma^2 + 1$ we get that as $|k| \to \infty$ $1 - |\omega s_1(\nu_k, \pi)| = 1 - |\omega \sin(arcctg\ \gamma)| + o(1) = 1 - \frac{|\omega|}{\sqrt{1+\gamma^2}} + o(1)$. Consequently, $\sigma_k = 1$ at sufficiently large values of $|k|$.

Sufficiency. Similar to the lemma in [7] it is easily proved that for the fucntion $d_j(z)$ constructed by formula (11) it holds the representation

$$d_j(z) = b_j \cos \pi z - a_j \sin \pi z + 2r + f_j, \tag{12}$$

where $f_j(z) = \int\limits_{-\pi}^{\pi} \tilde{f}_j(t) e^{itz} dt$, $\tilde{f}_j(t) \in L_2[-\pi, \pi]$. Since the fucntion

$$\sigma(z) = \frac{d_1(z) - d_2(z)}{\beta_1 - \beta_2} \tag{13}$$

by formula (12) has the form $\sigma(z) = \cos \pi z - \gamma \sin \pi z + \dfrac{f_1(z) - f_2(z)}{\beta_1 - \beta_2}$, then by [2] its zeros ν_k satisfy the asymptotic formulae

$$\nu_k = k + \frac{1}{\pi} arcctg\ \gamma + \tau_k, \quad \sum_{k=-\infty}^{\infty} \tau_k^2 < \infty . \tag{14}$$

Construct the fucntion

$$V_1(z) = \frac{\beta_1 d_2(z) - \beta_2 d_1(z)}{\beta_1 - \beta_2} - 2r . \tag{15}$$

By (12) we have

$$V_1(z)=\frac{1}{\beta_1-\beta_2}\left[\beta_1\left(b_2\cos\pi z-a_2\sin\pi z\right)-\beta_2\left(b_1\cos\pi z-a_1\sin\pi z\right)+\right.$$

$$\left.+\beta_1 f_2(z)-\beta_2 f_1(z)\right]=$$

$$=\frac{\beta_1(\beta_2-\gamma)-\beta_2(\beta_1-\gamma)}{\beta_1-\beta_2}\cos\pi z-\frac{\beta_1 a_2-\beta_2 a_1}{\beta_1-\beta_2}\sin\pi z+\frac{\beta_1 f_2(z)-\beta_2 f_1(z)}{\beta_1-\beta_2}=$$

$$=-\gamma\cos\pi z-\left(1+|\omega|^2\right)\sin\pi z+f_3(z), \tag{16}$$

where $f_3(z)=\int\limits_{-\pi}^{\pi}\tilde f_3(t)e^{itz}dt,\ \ \tilde f_3(t)\in L_2[-\pi,\pi]$.

By (11) and (13) it follows from the second and third conditions of the theorem that $\xi\sigma\left(c^-_{j,k}\right)\ \geq 0,\ \xi\sigma\left(c^+_{j,k}\right)\ \leq 0,\ \xi\sigma\left(c^-_{j,k+1}\right)\ \geq 0,...,$ where $\xi=-1$ or $\xi=1$. Therefore the arrangement of sequences $\left\{c^{\pm}_{1,k}\right\}$, $\{\nu_k\}$ are defined by the inequality

$$...\leq c^-_{1,k}\leq c^-_{2,k}\leq\nu_{2k}\leq c^*_{1,k}\leq c^*_{2,k}\leq\nu_{2k+1}\leq c^-_{1,k+1}\leq c^-_{2,k+1}\leq ...., \tag{17}$$

where $\nu_m<\nu_{m+1},\ m=0,\pm1,\pm2...$.

By formula (13) it holds $d_1(\nu_k)=d_2(\nu_k)$. Then we get from (15) $V_1(\nu_k)=d_j(\nu_k)-2r=c_k$. By inequality (10) $c_k\geq 2|\omega|$ or $c_k\leq -2|\omega|$. Hence and from inequalities (17) it follows that the signs of the terms of the sequence $\{c_k\}$ alternate. Using representation (16) and asymptotic formula (14), for sufficiently great values of $|k|$ we have

$$c_k=-\gamma\cos\pi\nu_k-\left(1+|\omega|^2\right)\sin\pi\nu_k+f_3(\nu_k)$$

$$=(-1)^{k+1}\left[\gamma\cos(arcctg\ \gamma)\cos\pi\tau_k-\gamma\sin(arcctg\ \gamma)\sin\pi\tau_k+\right.$$

$$\left.+\left(1+|\omega|^2\right)\sin(arcctg\ \gamma)\cos\pi\tau_k+\left(1+|\omega|^2\right)\cos(arcctg\ \gamma)\sin\pi\tau_k\right]+$$

$$+f_3(\nu_k)=(-1)^{k+1}\frac{1+\gamma^2+|\omega|^2}{\sqrt{1+\gamma^2}}+\eta_k,\ \ \sum_{k=-\infty}^{\infty}\eta_k^2<\infty\ . \tag{18}$$

Consequently, there exists such a number h_k that

$$c_k=2(-1)^{k+1}|\omega|\,ch\ h_k. \tag{19}$$

It is clear that

$$\sqrt{c_k^2-4|\omega|^2}=2\left|\omega\ sh\ h_k\right|\ . \tag{20}$$

Assume

$$V_2(z)=-\gamma\cos\pi z-\left(1-|\omega|^2\right)\sin\pi z+\theta(z), \tag{21}$$

where

$$\theta(z)=\sigma(z)\sum_{k=-\infty}^{\infty}\frac{\theta_k}{(z-\nu_k)\,\sigma'(\nu_k)},$$

$$\theta_k = \gamma \cos \pi \nu_k + \left(1 - |\omega|^2\right) \sin \pi \nu_k + 2 (-1)^{k+1} \sigma_k |\omega \; sh \; h_k| \; .$$

Using relations (14), (18), (20) and estimate $\sqrt{1+x} = 1 + O(x)$ $(x \to 0)$ we get $\theta_k = \dfrac{(-1)^k}{\sqrt{1+\gamma^2}} \left(\gamma^2 + 1 - |\omega|^2 - \sigma_k \left|\gamma^2 + 1 - |\omega|^2\right|\right) + r_k$, $\sum\limits_{k=-\infty}^{\infty} r_k^2 < \infty$. Since $|\omega|^2 < \gamma^2 + 1$ and $\sigma_k = 1$ for sufficiently great values of $|k|$ then $\sum\limits_{k=-\infty}^{\infty} \theta_k^2 < \infty$. Therefore the function $\theta(z)$ admits the representation

$$\theta(z) = \int_{-\pi}^{\pi} \tilde{\theta}(t) e^{itz} dt, \; \tilde{\theta}(t) \in L_2[-\pi, \pi] \tag{22}$$

(according to theorem 28 of the paper [8] and Paley-Wiener theorem [8, p.47]). It is easily seen that $\theta(\nu_k) = \theta_k$, consequently

$$V_2(\nu_k) = 2 (-1)^{k+1} \sigma_k |\omega \; sh \; h_k| \; . \tag{23}$$

Introduce the function

$$s(z) = \frac{1}{2|\omega|^2} [V_1(z) - V_2(z)] \; . \tag{24}$$

By (16), (21) and (22) for this fucntion it holds the representation

$$s(z) = -\sin \pi z + \int_{-\pi}^{\pi} \psi(t) e^{itz} dt, \; \psi(t) \in L_2[-\pi, \pi] \; .$$

Hence, by the paper [3] we get that the zeros λ_k $(k = 0, \pm 1, \pm 2, ...)$ of the fucntion $s(z)$ satisfy the asymptotic formula

$$\lambda_k = k + \alpha_k, \; \sum_{k=-\infty}^{\infty} \alpha_k^2 < \infty \tag{25}$$

assuming $z = \lambda_k$ in (24) and taking into account (19), (23)

$$s(\nu_k) = \frac{1}{|\omega|^2} [V_1(\nu_k) - V_2(\nu_k)] =$$

$$= \frac{1}{2|\omega|^2} \left[2 (-1)^{k+1} |\omega| \, ch \; h_k - 2 (-1)^{k+1} \sigma_k |\omega \; sh \; h_k|\right] =$$

$$= \frac{(-1)^{k+1}}{|\omega|} (1 - \sigma_k |th \; h_k|)$$

and since $|th \; h_k| < 1$, then $sign \; s(\nu_k) = (-1)^{k+1}$. Consequently, the zeros of the fucntion $\sigma(z)$ and $s(z)$ alternate. Besides, the sequences $\{\nu_k\}$ and $\{\lambda_k\}$ satisfy asymptotic formulae (14) and (25). Then by the paper [2] there exists a unique matrix-fucntion $\begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$ (where $p(x)\,,\ q(x)\in L_2[0,\pi]$)such that the sequences $\{\nu_k\}$ and $\{\lambda_k\}$ are the spectra of boundary value problems generated on $[0,\pi]$ by equation (1) (with this matrix-fucntion) and boundary conditions (7) and $y_1(0)=y_1(\pi)=0$.

We can easily be convinced that the characteristic fucntion of the constructed boundary value problem $D\left(\omega,\beta_j,\gamma\right)$ coincides with the fucntion $d_j(z)$. The theorem is proved.

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