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ON UNIFORM APPROXIMATIONS OF GENERALIZED ANALYTICAL FUNCTIONS BY GENERALIZED POLYNOMIALS

Abstract

The possibility of uniform approximation of analytical functions by generalized polynomials is proved, and the approximations were estimated subject to the degree of polynomial and the boundary of domain.

As is known that the polynomials were introduced by Faber in 1903 which later had been called Faber polynomials (see for example [2], p.114).

Let G be a bounded continuum containing more than one point and G_∞ such of adjacent domains which belongs to the point $z=\infty$. This is a simply-connected domain of extended surface whose domain Γ is the part of continuum G. Let's map conformally $w=0$ on the exterior of circle with the center at the point $w=\varphi(z)$ by virtue of the function $w=\Phi(z)$. We require that the following two conditions

$$\Phi(\infty)=\infty \quad \text{and} \quad \lim_{z\to\infty}\frac{\Phi(z)}{z}=1 \tag{1}$$

will be satisfied.

Hence it follows that in the neighborhood of the point $z=\infty$

$$\Phi(z)=z+\alpha_0+\frac{\alpha_{-1}}{z}+...$$

$$[\Phi(z)]^n=z^n+\alpha_{n-1}^{(n)}z^{n-1}+...+\alpha_0^{(n)}+\frac{\alpha_{-1}^{(n)}}{z}+...$$

The polynomials

$$\Phi_n(z)=z^n+\alpha_{n-1}^{(n)}z^{n-1}+...+\alpha_0^{(n)} \tag{2}$$

representing the totality of members with non-negative degrees z in Laurent extension of functions $[\Phi(z)]^n$ are called Faber polynomials generated by the continuum G.

Every function $f(z)$ regular on the bounded, closed set $\bar{G}$ whose complement is a simply connected domain can be expanded in series by Faber polynomials (see [2] p.118)

$$f(z)=\sum_{n=0}^{\infty}a_n\Phi_n(z), \quad z\in\bar{G}, \tag{3}$$

where

$$a_n = \frac{1}{2\pi i} \int\limits_{|w|=R} f(\psi(w)) \frac{dw}{w^{k+1}} = \frac{1}{2\pi i} \int\limits_{C_R} \frac{f(z)\,\Phi'(z)\,dz}{[\Phi(z)]^{n+1}}$$

Here $\psi(w) = \Phi^{-1}(w)$, C_2 is a circular image (level line), i.e., the pre-image of the circle $|w| = R$ at mapping $w = \varphi(z)$ The Faber polynomials are generalized for the case of generalized analytical functions.

Consider the class of generalized analytical functions $U_{p,2}(A, B, G)$ in the sense of I.N. Vekua, i.e., the class of regular solutions of the equation (see [1], p.156)

$$\underset{\bar{z}}{\partial} F(z) + A(z) F(z) + B(z) \bar{F}(z) = 0, \tag{4}$$

where $A(z),\ B(z) \in \mathcal{L}_p(G),\ p > 2,\ \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$

If the generalized analytical function in the extended surface has a unique pole $z = \infty$ then we'll call it the generalized polynomial (see [1], p.167). In such case the order of pole we'll call the degree of generalized polynomial. In [1] (p.202) the construciton of a generalized polynomial is given. It was proved there that if $F(z) \in U_{p,2}(A, B, G)$ and $F(z) \in C(\bar{G})$ (continuous in $\bar{G}$), then (see [1], p.194)

$$F(z) \equiv K(f, G) = \frac{1}{2\pi i} \int\limits_{\Gamma} \Omega_1(z,t) f(t)\, dt - \Omega_2(z,t) \overline{f(t) dt}, \tag{5}$$

where $\Omega_1(z,t)$ and $\Omega_2(z,t)$ are normed kernels of the class $U_{p,2}(A, B, G)$, and

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{F(t)\, dt}{t - z}, \tag{6}$$

is analytical in G and continuous on $\bar{G}\,(\Gamma = \partial G)$.

Now let $\{\Phi_n(z)\}$ be Faber polynomials be generated by the continuum $\bar{G}$.

Let's to the system $\{\Phi_n(z)\}$ by formula (5) take the system $\{\Phi_k(z)\}$ by the following form

$$\Phi_{2k}(z) = K(\Phi_k(z), G), \qquad \Phi_{2k+1}(z) = K(i\Phi_k, G) \tag{7}$$

Note that $\Phi_{2k}(z)$ and $\Phi_{2k+1}(z)$ are generalized polynomials of degree k.

The polynorms $\Phi_k(z)$ we'll call Faber generalized polynomial for the class $U_{p,2}(A, B, G)$ generated by $\bar{G}$.

In particular at $A = B = 0$, $F(z)$ turn into analytical in G functions, and polynomials $\{\Phi_k(z)\}$ coincide with Faber polynomials $[\Phi_k(z)]$.

Theorem 1. *Let $F(z) \in U_{p,2}(A, B, G)$ and $F(z) \in C(\bar{G})$. Then we can represent it in the form of uniformly convergent in $\bar{G}$ series*

$$F(z) = \sum_{k=0}^{\infty} C_{2k}\Phi_{2k}(z) + C_{2k+1}\Phi_{2k+1}(z), \tag{8}$$

where

$$C_{2k}+iC_{2k+1}=\frac{1}{2\pi i}\int\limits_{|w|=R}\frac{F(\psi(w))}{w^{k+1}}dw.$$

Proof. Note that by formula (6) on the boundary $F(t)=f(t)$ almost everywhere, since $f(t)$ is represented by the Cauchy formula (Γ is closed and $f(z)$ is analytical in G and continuous on $\bar{G}$).

Consider the following expression

$$\mathcal{P}_n(z)=\sum_{k=0}^{n}C_{2k}\Phi_{2k}(z)+C_{2k+1}\Phi_{2k+1}(z)$$

It is clear that $\mathcal{P}_n(z)$ is a generalized polynomial of degree n. Now we'll establish that $\mathcal{P}_n(z)=K(f_n,G)$, where

$$f_n(z)=\sum_{k=0}^{n}a_k\Phi_k(z).$$

We have:

$$K(f_n,G)=\frac{1}{2\pi i}\int\limits_{\Gamma}\Omega_1(z,t)\sum_{k=0}^{n}a_k\Phi_k(t)-\Omega_2(z,t)\overline{\sum_{k=0}^{n}a_k\Phi_k(t)}\overline{dt}=$$

$$=\frac{1}{2\pi i}\int\limits_{\Gamma}\Omega_1(z,t)\sum_{k=0}^{n}(C_{2k}+iC_{2k+1})\Phi_k(t)-$$

$$-\Omega_2(z,t)\overline{\sum_{k=0}^{n}(C_{2k}+iC_{2k+1})\Phi_k(t)}\overline{dt}=$$

$$=\sum_{k=0}^{n}\left[\frac{1}{2\pi i}\int\limits_{\Gamma}\Omega_1(z,t)\Phi_k(t)\,dt-\Omega_2(z,t)\overline{F_k(t)}\overline{dt}\right]C_{2k}+$$

$$+\sum_{k=0}^{n}\left[\frac{1}{2\pi i}\int\limits_{\Gamma}\Omega_1(z,t)(i\Phi_k(t))\,dt-\Omega_2(z,t)\left(\overline{i\Phi_k(t)\,dt}\right)\right]C_{2k+1}=$$

$$=\sum_{k=0}^{n}C_{2k}\Phi_{2k}(z)+C_{2k+1}\Phi_{2k+1}(z)=\mathcal{P}_n(z).$$

Now let's consider the difference $F(z)-\mathcal{P}_n(z)$ and estimate it

$$|F(z)-\mathcal{P}_n(z)|=\left|\left[\frac{1}{2\pi i}\int\limits_{\Gamma}\Omega_1(z,t)\left(f(t)-\sum_{k=0}^{n}a_k\Phi_k(t)\,dt\right)-\right.\right.$$

$$\left.\left.-\Omega_2(z,t)\overline{\left(f(t)-\sum_{k=0}^{n}a_k\Phi_k(t)\right)\overline{dt}}\right]\right|\le$$

$$\leq \frac{1}{2\pi}\int\limits_{\Gamma} \left(\left|\Omega_1(z,t)\right| + \left|\Omega_2(z,t)\right|\right) \left|f(t) - \sum_{k=0}^{n} a_k \Phi_k(t)\right| |dt|,$$

since

$$\left|f(t) - \sum_{k=0}^{n} a_k \Phi_k(t)\right| = \left|\overline{f(t) - \sum_{k=0}^{n} a_k \Phi_k(t)}\right| \qquad \text{and} \qquad |dt| = \left|\overline{dt}\right|.$$

By virtue of the fact that at $z \in K, K \subset G$ is a compact, $(|\Omega_1| + |\Omega_2|)$ is bounded (see [1], p.205) and $\left|f(z) - \sum\limits_{k=0}^{n} a_k \Phi_k(z)\right| \to 0$ as $n \to \infty$ then we have:

$$|F(z) - \mathcal{P}_n(z)| \to 0 \quad \text{as} \quad n \to \infty$$

i.e.

$$F(z) = \sum_{k=0}^{\infty} C_{2k} \Phi_{2k}(z) + C_{2k+1} \Phi_{2k+1}(z)$$

The theorem is proved.

Now let $F(z) \in U_{p,2}(A, B, G)$ and $F(z) \in C\left(\bar{G}\right)$

Then by formula (5)

$$F(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \Omega_1(z,t) f(t)\, dt - \Omega_2(z,t)\, \overline{f(t)}\overline{dt},$$

where

$$f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{F(t)\, dt}{t - z} \tag{9}$$

is analytical in G and continuous on $\bar{G}$.

Theorem 2. *Let $\partial G = \Gamma$ be a smooth curve. $F(z) \in U_{p,2}(A, B, G)$ and $F(z) \in C\left(\bar{G}\right)$. If the function $f(z)$ determined by formula (9) at some $s \in (0,1)$ satisfies the condition $f\left(\psi\left(e^{i0}\right)\right) \in Lip_M S$ then at each natural $n = 1, 2, ...$ there exist the generalized polynomials $P_n(z)$ of degree $\leq n$ that at all $z \in \bar{G}$ the inequalities*

$$|F(z) - \mathcal{P}_n(z)| \leq \frac{N \cdot M}{n^s}$$

are satisfied, where N is a constant not depending on F, n^s and z.

Proof. Since analytical in G continuous on $\bar{G}$ function $f(z)$ satisfies the conditions of V.K. Dzyadik, Yo.I. Volkov and G.A. Alibekov theorem (see [3], p.460) there exists the polynomial $P_n(z)$ of degree $\leq n$, such that at all $z \in \bar{G}$ the inequality

$$|f(z) - P_n(z)| \leq \frac{A \cdot M}{n^s}. \tag{10}$$

is fulfilled.

Denote by $P_n(z)$ the coefficient of polynomial a_n and introduce the following notion

$$C_{2n} + iC_{2n+1} = a_n, \quad n = 0, 1, 2, ...$$

$$P_n(z) = \sum_{k=0}^{\infty} a_k z^k$$

Construct the generalized polynomials of the form

$$\mathcal{P}_n(z) = \sum_{k=0}^{\infty} C_{2k}\Phi_{2k}(z) + C_{2k+1}\Phi_{2k+1},$$

where $\Phi_k(z)$ are generalized Faber polynomials determined above.

We were convinced that

$$\mathcal{P}_n(z) = K(f_n, G), \quad \text{where} \quad f_n(z) = \sum_{k=0}^{n} a_k z^k$$

Representing the function $F(z) \in U_{p,2}(A, B, G)$ in the form

$$F(z) = \frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z,t) f(t)\, dt - \Omega_2(z,t)\overline{f(t)}\overline{dt}$$

and considering the difference $F(z) - \mathcal{P}_n(z)$ estimating it as above we finally have:

$$|F(z) - \mathcal{P}_n(z)| \le \frac{1}{2\pi}\int_{\Gamma} (|\Omega_1| + |\Omega_2|)\, |f(t) - P_n(t)|\, |dt|$$

Allowing for $|f(z) - P_n(z)| \le \frac{A \cdot M}{n^s}$ by virtue of (10) and

$$\int_{\Gamma} (|\Omega_1| + |\Omega_2|)\, |dt| \le M_1 \cdot \int_{\Gamma} |dt| = M_1 \quad l, z \in k, \quad K \subset G \text{ is a compact}$$

where l is the length of Γ we have:

$$|F(z) - \mathcal{P}_n(z)| \le N \cdot M \frac{1}{n^s}, \quad \forall z \in \bar{G}$$

The theorem is proved.

References

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