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ON CLOSURE OF SUM OF TWO ALGEBRAS OF REAL CONTINUOUS FUNCTIONS

Abstract

In this paper we cite the proof of the Stone-Weierstrass theorem in general case – without requirement of disappearance and separation of points with respect to some subalgebra of the space $C_R(K)$, where K is a compact set. Besides, the closure problem of sum of two subalgedras of the algebra $C_R(K)$ is studied.

The Stone-Weierstrass approximation theorem in the algebra of $C_R(K)$ is well-known ([1] p.296). In the suggested paper we cite another proof of this theorem in general case, i.e. without requirement of disappearance and separation of points (of compact) with respect to some subalgebra of the algebra of $C_R(K)$. Besides the case when the studied family is sum of two subalgebras, is considered.

1. Stone-Weierstrass type theorem.

Let K be a compact topological space. Denote by $C_R(K)$ a space of real continuous functions on compact K with the norm $\|f\| = \sup\limits_{x\in K} |f(x)|$.

Let A be a subalgebra of the algebra $C_R(K)$. The algebra A generates the equivalence relation determined by the following form

$$x \sim y \overset{def}{\equiv} \forall f \in A : f(x) = f(y).$$

This relation divides the set K into disjoint classes

$$\xi \overset{def}{\equiv} [x]_A = \{y \in K \mid \forall f \in A : f(y) = f(x)\}.$$

Denote $\tilde{K} = \{\xi = [x]_A \mid x \in K\}$ and consider the project function $p : K \to \tilde{K}$ defined by the equality $p(x) = [x]_A$. On the set $\tilde{K}$ we define the quotient topology in the following way: we call the set $\tilde{G} \subset \tilde{K}$ an open set if $p^{-1}(G)$ is open in K. It is known [2, p.127] that $p : K \to \tilde{K}$ is a continuous function, and $\tilde{K}$ as image of continuous function is a compact space.

We associate the function $\tilde{f} : \tilde{K} \to R$ defined by the equality

$$\tilde{f}([x]_A) = f(x) \tag{1.1}$$

to each function and $f \in A$.

It is obvious that $f = \tilde{f} \circ p$ and $\tilde{f}$ is a continuous function [2,p.134]: $\tilde{f} \in C_R\left(\tilde{K}\right)$. Denote by $[A]$ a set of functions $\tilde{f} : \tilde{K} \to R$ of (1.1). The set $[A]$ is algebra, A and $[A]$ are isometric $\left\|\tilde{f}\right\|_{C_R(\tilde{K})} = \|f\|_{C_R(K)}$.

Denote

$$\xi_0 = [x]_A^0 = \{x \mid \forall f \in A : f(x) = 0\}.$$

If algebra A won't disappear at any point of the set K, i.e. for any $x \in K\ \exists f \in A$ such that $f(x) \neq 0$, then $\xi_0 = \varnothing$.

At first we consider the case $\xi_0 = \varnothing$.

Lemma 1.1. *For* $\tilde{f} \in [A]$ *it holds* $\tilde{f}(\xi_0) = 0$.

The proof is obvious, since if $x \in \xi_0$ and $f = \tilde{f} \circ p$, then $\tilde{f}(\xi_0) = f(x) = 0$..

Lemma 1.2. *If* $\xi \neq \xi_0 \quad \left(\xi \in \tilde{K}\right)$, *then there exists the function* $\tilde{f} \in [A]$, *such that* $\tilde{f}(\xi) \neq 0$.

Proof. Let $x \in \xi$. Since $x \notin \xi_0$, then there exists the function $f \in A$ such that $f(x) \neq 0$. Then $\tilde{f}(\xi) = \tilde{f}([x]_A) = f(x) \neq 0$.

Lemma 1.3. *The algebra* $[A]$ *divides the points of the set* $\tilde{K}$, *i.e.* $\forall \xi, \eta \in \tilde{K}$ $(\xi \neq \eta)$, $\exists \tilde{f} \in [A]$ *such that* $f(\xi) \neq f(\eta)$.

Proof. Let $x \in \xi$ and $y \in \eta$. Since $\xi \neq \eta$, then $x \not\sim y$ (x and y are not equivalent). Then there exists the function $f \in A$ such that $f(x) \neq f(y)$. Hence, $\tilde{f}(\xi) = f(x) \neq f(y) = \tilde{f}(\eta)$.

Lemma 1.4. *For any* $\xi, \eta \in \tilde{K} \setminus \{\xi_0\}$ $(\xi \neq \eta)$ *there exist the functions* $\tilde{w}_{\xi\eta}, \tilde{w}_{\eta\xi} \in [A]$ *such that*

$$\tilde{w}_{\xi\eta}(\xi) = 1, \tilde{w}_{\xi\eta}(\eta) = 0 \quad \textit{and} \quad \tilde{w}_{\eta\xi}(\xi) = 0, \tilde{w}_{\eta\xi}(\xi) = 0.$$

The proof is analogously to theorem 7.29 from [3] (p.182).

Lemma 1.5. *For any* $\xi, \eta \in \tilde{K} \setminus \{\xi_0\}$ $(\xi \neq \eta)$ *and for any real numbers* $a, b \in R$ *there exists the function* $\tilde{f} \in [A]$ *such that* $\tilde{f}(\xi) = a$ *and* $\tilde{f}(\eta) = b$.

The proof follows from lemma 1.4 if we assume

$$\tilde{f} = a\ \tilde{w}_{\xi\eta}(\xi) + b\tilde{w}_{\eta\xi}(\xi).$$

Denote

$$C_R^{A,0}\left(\tilde{K}\right) = \left\{\ \tilde{f}\ \in C_R\left(\tilde{K}\right) \middle|\ \tilde{f}(\xi_0) = 0\right\}.$$

Theorem 1.1. $\overline{[A]} = C_R^{A,0}\left(\tilde{K}\right)$, *where* $\overline{[A]}$ *is a closure of the algebra* $[A]$ *by norm of the space* $C_R\left(\tilde{K}\right)$.

The proof follows from the fact that for the algebra $[A]$ all the conditions of Stone-Weierstrass theorem are satisfied.

Corollary 1.1. *If* $\xi_0 = \varnothing$, *then* $\overline{[A]} = C_R\left(\tilde{K}\right)$.

Let's introduce the following notation:

$$C_R^A(K) = \left\{\ f\ \in C_R(K) |\ f|_{[x]_A} \equiv f(x)\right\}$$

where $f|_M$ is contraction of the function $f\ \in C_R(K)$ to the subset $M \subset K$. Further, if $\xi_0 = [x]_A^0 = \varnothing$, then we assume

$$C_R^{A,0}(K) = \left\{\ f\ \in C_R(K) |\ f|_{[x]_A^0} \equiv 0\right\},$$

and if $\xi_0 = [x]_A^0 = \varnothing$, then we shall assume that $C_R^{A,0}(K) = C_R^A(K)$. Denote

$$E_R^A(K) = C_R^A(K) \cap C_R^{A,0}(K). \tag{1.2}$$

Theorem 1.2. *Let A be some subalgebra of the algebra $C_R(K)$. Then $\bar{A} = E_R^A(K)$, where $\bar{A}$ is closure of subalgebra A by norm of the space $C_R(K)$.*

Proof. Let $f \in E_R^A(K)$ and $\varepsilon > 0$ be any positive number. Consider the function $\tilde{f} : \tilde{K} \to R : \tilde{f}([x]_A) = f(x)$. Since $f \in E_R^A(K)$, then it is clear that $\tilde{f} \in C_R^{A,0}\left(\tilde{K}\right)$ (we consider the case $\xi_0 \neq \varnothing$, the case $\xi_0 = \varnothing$ is considered analogously). Then by theorem 1.1 there exists the function $\tilde{g} \in [A]$ such that $\left\|\tilde{f} - \tilde{g}\right\|_{C_R(\tilde{K})} < \varepsilon$. Hence for the function $g = \tilde{g} \circ p \in A$ and $\forall x \in K$ we have

$$|f(x) - g(x)| = \left|\tilde{f}([x]_A) - \tilde{g}([x]_A)\right| \leq \left\|\tilde{f} - \tilde{g}\right\|_{C_R(\tilde{K})} < \varepsilon.$$

Theorem 1.2 is proved.

Note that theorem 1.2 when K is a compact on plane, is proved in the paper [4].

2. On closure of sum of two subalgebras of the algebra $C_R(K)$.

Let the sets of the functions A_1 and A_2 be subalgebras of the algebra of real continuous functions of $C_R(K)$. Denote by A the algebraic sum of A_1 and A_2 : $A = A_1 + A_2$. It is clear that A, in general, is not algebra.

Let $E_R^{A_i}(K) \subset C_R(K)$ $\left(i, j = \overline{1,2}\right)$ be the sets of the functions defined according to(1.2) with respect to algebras A_i $\left(i = \overline{1,2}\right)$.

Theorem 2.1.

$$\bar{A} = \overline{E_R^{A_1}(K) + E_R^{A_2}(K)}, \tag{2.1}$$

where the closure is taken by norm of the space $C_R(K)$.

Proof. It is easy to see that the following chain of inclusions is valid

$$A_1 + A_2 \subset \bar{A}_1 + \bar{A}_2 \subset \overline{A_1 + A_2}, \tag{2.2}$$

By theorem 1.2 $\bar{A}_1 = E_R^{A_1}(K)$ and $\bar{A}_2 = E_R^{A_2}(K)$. Then in (2.2) taking the closure we obtain (2.1). The theorem is proved.

We introduce the equivalence relation on the compact set K by the following form

$$x \overset{A}{\sim} y \overset{def}{\equiv} \forall f \in A : f(x) = f(y) \Longleftrightarrow$$

$$\Longleftrightarrow \forall f_1 \in A_1 \wedge \forall f_2 \in A_2 : f_1(x) + f_2(x) = f_1(y) + f_2(y).$$

Denote $\xi = [x]_A \overset{def}{\equiv} \{y \in K \mid \forall f \in A : f(y) = f(x)\}$.

Let $[x]_{A_i} \overset{def}{\equiv} \{y \in K \mid \forall f_i \in A_i : f_i(y) = f_i(x)\}$ $\left(i = \overline{1,2}\right)$.

Lemma 2.1.

$$[x]_A = [x]_{A_1} \cap [x]_{A_2}.$$

Proof. Let $y\in[x]_{A_1}\cap[x]_{A_2}$. Then

$$\forall f_1\in A_1\wedge\forall f_2\in A_2: f_1(x)=f_1(y)\wedge f_2(x)\Longrightarrow$$
$$\Longrightarrow f_1(x)+f_2(x)=f_1(y)+f_2(y)\Longrightarrow y\in[x]_A\Longrightarrow[x]_{A_1}\cap[x]_{A_2}\subset[x]_A.$$

Let $y\in[x]_A$, then

$$y\in[x]_A\Longrightarrow\forall f_1\in A_1\wedge\forall f_2\in A_2: f_1(x)+f_2(x)=f_1(y)+f_2(y)\Longrightarrow$$
$$\Longrightarrow\forall f_1\in A_1\wedge\forall f_2\equiv 0\in A_2:\ f_1(x)=f_2(x)\Longrightarrow y\in[x]_{A_1}.$$

Analogously $y\in[x]_{A_2}$, so $y\in[x]_{A_1}\cap[x]_{A_2}$, i.e. $[x]_A\subset[x]_{A_1}\cap[x]_{A_2}$. The lemma is proved.

Analogously, as in point 1, in the set $\tilde{K}=\{\xi=[x]_A|\ x\in K\}$ we introduce the quotient topology and respectively for each function $f\in A=A_1+A_2$ we define the function

$$\tilde{f}(\xi)=\tilde{f}([x]_A)=f(x).$$

It is obvious that $\tilde{f}=f\circ p$, where $p(x)=[x]_A$ and $\left\|\tilde{f}\right\|_{C_R(\tilde{K})}=\|f\|_{C_R(\bar{K})}$. If $f=f_1+f_2$ $(f_1\in A_1,\ f_2\in A_2)$, then $\tilde{f}=\tilde{f}_1+\tilde{f}_2$, where $\tilde{f}_1([x]_A)=f_1(x)$ and $\tilde{f}_2([x]_A)=f_2(x)$.

It is clear that under these considerations the linear system of the function A generates the linear system of the functions $[A]\subset C_R\left(\tilde{K}\right)$, and the subalgebras $A_i,\ i=\overline{1,2}$ generate the algebras $[A_i]\subset C_R\left(\tilde{K}\right)$, $i=\overline{1,2}$, respectively, moreover $[A]=[A_1]\dot{+}[A_2]$. There exists one-to-one correspondence between the linear systems A and $[A]$ (between the algebras A_i and $[A_i]$, $i=\overline{1,2}$, respectively).

Let $\xi_0\overset{def}{\equiv}[x]_A^0=\{x\in K\mid\forall f\in A: f(x)=0\}$.

It is obvious that $\xi_0=\xi_0^1\cap\xi_0^2$, where

$$\xi_0^1=\{x\in K\mid\forall f_1\in A: f_1(x)=0\}$$

and

$$\xi_0^2=\{x\in K\mid\forall f_2\in A: f_2(x)=0\}.$$

Assume (at first we assume that $\xi_0\neq\varnothing$):

$$\hat{K}_1=\left\{\ \xi_1=[x]_{A_1}\ \mid x\in K\right\},\ \hat{K}_2=\left\{\ \xi_2=[x]_{A_2}\ \mid x\in K\right\},$$
$$\hat{K}=\left\{\ \xi=\xi_1\cap\xi_2|\ \xi_1\in\hat{K}_1\backslash\left\{\xi_0^1\right\},\ \xi_2\in\hat{K}_2\backslash\left\{\xi_0^2\right\}\right\},$$
$$\hat{K}_1^0=\left\{\ \xi=\xi_1^0\cap\xi_2|\ \xi_2\in\hat{K}_2\backslash\left\{\xi_2^0\right\}\right\},$$
$$\hat{K}_2^0=\left\{\ \xi=\xi_1\cap\xi_2^0|\ \xi_1\in\hat{K}_1\backslash\left\{\xi_1^0\right\}\right\}.$$

The sets $\hat{K}$, $\hat{K}_1^0$ and $\hat{K}_2^0$ do not intersect and

$$\tilde{K}=\hat{K}\cup\ \hat{K}_1^0\cup\hat{K}_2^0\cup\{\xi_0\}.$$

The next lemma follows obviously from the construction of the sets $\hat{K}$, $\hat{K}_1^0$ and $\hat{K}_2^0$.

Lemma 2.2.

(a) For any $\xi \in \hat{K}$ *there exist the functions* $\tilde{h}_1 \in [A_1]$ *and* $\tilde{h}_2 \in [A_2]$ *such that* $\tilde{h}_1(\xi) \neq 0,\ \tilde{h}_2(\xi) \neq 0$*;*

(b) For any $\xi \in \hat{K}_1^0$ *and for any* $\tilde{h}_1 \in [A_1] : \tilde{h}_1(\xi) = 0$*;*

(c) For any $\xi \in \hat{K}_1^0$ *there exists* $\tilde{h}_2 \in [A_2] : \tilde{h}_2(\xi) = 0$*;*

(d) For any $\xi \in \hat{K}_2^0$ *and for any* $\tilde{h}_2 \in [A_2] : \tilde{h}_2(\xi) = 0$*;*

(e) For any $\xi \in \hat{K}_2^0$ *there exists* $\tilde{h}_1 \in [A_1] : \tilde{h}_1(\xi) = 0$.

Lemma 2.3. *The system of the functions* $[A]$ *separates the points of the set* $\tilde{K}$ *, i.e. for any* $\xi, \eta \in \tilde{K}$ $(\xi \neq \eta)$ *there exists the function* $\tilde{f} \in [A]$ *such that* $\tilde{f}(\xi) \neq \tilde{f}(\eta)$.

The proof is analogous to the proof of lemma 1.3.

Lemma 2.4. *For any* $\xi, \eta \in \tilde{K}$ $(\xi \neq \eta)$ *at least of one of the following conditions is satisfied*

1) $\exists \tilde{g}_1 \in [A_1]$ *for which* $\tilde{g}_1(\xi) \neq \tilde{g}_1(\eta)$*;*

2) $\exists \tilde{g}_2 \in [A_2]$ *for which* $\tilde{g}_2(\xi) \neq \tilde{g}_2(\eta)$.

Proof. Nonfulfilment of the both conditions 1) and 2) should mean for any function $\tilde{g} = \tilde{g}_1 + \tilde{g}_2 \in [A]$ $(\tilde{g}_1 \in [A_1]$ and $\tilde{g}_2 \in [A_2])$, and $\tilde{g}(\xi) = \tilde{g}_1(\xi) + \tilde{g}_2(\xi) = \tilde{g}_1(\eta) + \tilde{g}_2(\eta) = \tilde{g}(\eta)$, that is impossible by lemma 2.4.

Lemma 2.5. *For any* $\xi, \eta \in \tilde{K} \setminus \{\xi_0\}$ $(\xi \neq \eta)$ *there exist the functions* $\tilde{w}_{\xi\eta}, \tilde{w}_{\eta\xi} \in [A]$ *such that* $\tilde{w}_{\xi\eta}(\xi) = 1, \tilde{w}_{\xi\eta}(\eta) = 0$ *and* $\tilde{w}_{\eta\xi}(\eta) = 0, \tilde{w}_{\eta\xi}(\eta) = 1$.

Proof. We shall say that for the point ξ and algebra $[A_i]$ the condition $(*)$ is satisfied, if there exist the functions $\tilde{g}_i, \tilde{h}_i \in [A_i]$ such that $\tilde{g}_i(\xi) \neq \tilde{g}_i(\eta)$ and $\tilde{h}_i(\xi) \neq 0$. Note that if condition $(*)$ is satisfied, then following the method of the proof of theorem 7.29 from [3] (p.182) we can show that there exists the function $\tilde{w}_{\xi\eta} \in [A_i]$ such that $\tilde{w}_{\xi\eta}(\xi) = 1$ and $\tilde{w}_{\xi\eta}(\eta) = 0$.

Consider the following cases

(a) $\xi, \eta \in \hat{K}$;

(b) $\xi, \eta \in \hat{K}_1^0$;

(c) $\xi, \eta \in \hat{K}_2^0$;

(d) $\xi \in \hat{K}_1^0,\ \eta \in \hat{K}_2^0$;

(e) $\xi \in \hat{K},\ \eta \in \hat{K}_1^0$;

(f) $\xi \in \hat{K},\ \eta \in \hat{K}_2^0$.

The cases (c) and (f) are analogous to the cases (b) and (e), respectively, therefore we won't consider them.

In the case (a) by lemmas 2.2 (a) and 2.4 for the points ξ, η of the algebra A_1 (or the algebra A_2) the condition $(*)$ is satisfied, therefore there exist the desired functions $\tilde{w}_{\xi\eta}$ and $\tilde{w}_{\eta\xi}$.

In the case (b) by the lemma 2.2 (b) $\forall \tilde{h}_1 \in [A_1]$ and $\tilde{h}_1(\xi) = \tilde{h}_1(\eta) = 0$ holds. This time by lemmas 2.4 and 2.2(c) there exist the functions $\tilde{g}_2, \tilde{h}_2, \tilde{h}_2' \in [A_2]$ such that $\tilde{g}_2(\xi) \neq \tilde{g}_2(\eta)$, $\tilde{h}_2(\xi) \neq 0$ and $\tilde{h}_2'(\eta) \neq 0$. Then for the points ξ, η and algebra $[A_2]$ the condition $(*)$ is satisfied, therefore there exist the desired functions $\tilde{w}_{\xi\eta}, \tilde{w}_{\eta\xi} \in [A_2]$.

In the case (d) from lemma 2.2 (b)-(e) it follows that there exist the functions $\tilde{h}_1 \in [A_1]$ and $\tilde{h}_2 \in [A_2]$ such that $\tilde{h}_1(\xi) = 0$, $\tilde{h}_1(\eta) \neq 0$ and $\tilde{h}_2(\xi) \neq 0$, $\tilde{h}_2(\eta) = 0$. Then $\tilde{w}_{\xi\eta}(t) = \tilde{h}_2(t)/\tilde{h}_2(\xi)$ and $\tilde{w}_{\eta\xi}(t) = \tilde{h}_1(t)/\tilde{h}_1(\xi)$ will be desired functions.

In the case (e) by lemma 2.2 (a)-(c) there exists the function $\tilde{h}_1 \in [A_1]$ such that $\tilde{h}_1(\xi) \neq 0$ and $\tilde{h}_1(\eta) = 0$. Then the function $\tilde{w}_{\xi\eta}(t) = \tilde{h}_1(t)/\tilde{h}_1(\xi)$ satisfies the conditions $\tilde{w}_{\xi\eta}(\xi) = 1, \tilde{w}_{\xi\eta}(\eta) = 0$.

For the construction of the function $\tilde{w}_{\eta\xi}$ we consider two cases:

1) $\exists \tilde{g}_2 \in [A_2]$ such that $\tilde{g}_2(\xi) \neq \tilde{g}_2(\eta)$. By lemma 2.2(c) $\exists\ \tilde{h}_2 \in [A_2]$ such that $\tilde{h}_2(\eta) \neq 0$. The point η and algebra $[A_2]$ satisfy the condition $(*)$, i.e. there is the required function $\tilde{w}_{\eta\xi} \in [A_2]$.

2) for any function $\tilde{g}_2 \in [A_2] : \tilde{g}_2(\xi) \neq \tilde{g}_2(\eta)$. Then by lemma 2.4 $\tilde{g}_1 \in [A_1]$ such that $\tilde{g}_1(\xi) \neq 0$ $(\tilde{g}_2(\eta) = 0)$ and $\exists \tilde{h}_2 \in [A_2] : \tilde{h}_2(\eta) \neq 0$. Then

$$\tilde{w}_{\eta\xi}(t) = -\frac{\tilde{h}_2(\xi)}{\tilde{h}_2(\eta)\,\tilde{g}_1(\xi)}\tilde{g}_1(t) + \frac{\tilde{h}_2(t)}{\tilde{h}_2(\eta)} \in [A]$$

will be a desired function.

The lemma is proved.

Lemma 2.6. *For any $\xi, \eta \in \tilde{K} \setminus \{\xi_0\}$ $(\xi \neq \eta)$ and for any real numbers $a, b \in R$ there exists the function $\tilde{f} \in [A]$ such that $\tilde{f}(\xi) = a$ and $\tilde{f}(\eta) = b$.*

The proof follows from lemma 2.5 if we assume $\tilde{f} = a\tilde{w}_{\xi\eta} + b\tilde{w}_{\eta\xi}$, then this will be a desired function.

Theorem 2.2. *If the closure of sets $[A]$ is a lattice, then $[A] = C_R^{A,0}\left(\tilde{K}\right)$, where $C_R^{A,0}\left(\tilde{K}\right) = \left\{ f \in C_R\left(\tilde{K}\right) \middle| f(\xi_0) = 0 \right\}$.*

Proof. If $\overline{[A]}$ is a lattice, then it follows from $\tilde{f}_1, ..., \tilde{f}_n \in \overline{[A]}$ that

$$\max\left\{\tilde{f}_1, ..., \tilde{f}_n\right\} \in \overline{[A]}, \quad \min\left\{\tilde{f}_1, ..., \tilde{f}_n\right\} \in \overline{[A]}.$$

Further, by lemma 2.6 $\forall f \in C_R^{A,0}\left(\tilde{K}\right)$ and $\forall \xi, \eta \in \tilde{K}$, $\exists \tilde{h}_{\xi\eta}(\xi) \in A$ and such that $\tilde{h}_{\xi\eta}(\xi) = \tilde{f}(\xi)$ and $\tilde{h}_{\xi\eta}(\eta) = \tilde{f}(\eta)$. Further, the proof is led analogously to the proof of the Stone-Weierstrass theorem (see [3], p.183).

Denote by

$$C_R^A(K) = \left\{ f \in C_R(K) \middle| \ f|_{[x]_A} \equiv f(x) \right\}.$$

If $\xi_0 = [x]_A^0 \neq \varnothing$, then we assume

$$C_R^{A,0}(K) = \left\{ f \in C_R(K) \middle| \ f|_{[x]_A} \equiv 0 \right\},$$

if $\xi_0 = [x]_A^0 = \varnothing$, then we shall assume that $C_R^{A,0}(K) = C_R^A(K)$.

Denote

$$E_R^A(K) = C_R^A(K) \cap C_R^{A,0}(K).$$

Theorem 2.2 yields:

Theorem 2.3. *Let A_1 and A_2 be subalgebras in $C_R(K)$. If the closure $\bar{A} \equiv \overline{A_1 + A_2}$ in $C_R(K)$ is a lattice, then $\bar{A} = E_R^A(K)$.*

The next one immediately follows from this theorem.

Theorem 2.4. *Let A_1 and A_2 be subalgebras in $C_R(K)$. In order $\bar{A} \equiv A_1 + A_2 = E_R^A(K)$ it is necessary and sufficient that $\bar{A}$ be algebra.*

Remark. From theorem 2.1 it follows that $\overline{A_1 + A_2} = E_R^{A_1}(K) + E_R^{A_2}(K)$. In general, $\overline{E_R^{A_1}(K) + E_R^{A_2}(K)} \neq E_R^A(K)$. Indeed, let $K = [a, b]$ and $a < c < d < b$. Assume

$$A_1 = E_R^{A_1}(K) = \left\{ f \in C_R[a,b] \,\middle|\, f|_{[a,c]} \equiv const \text{ and } f|_{[d,b]} \equiv const \right\},$$
$$A_2 = E_R^{A_2}(K) = \left\{ f \in C_R[a,b] \,\middle|\, f|_{[c,d]} \equiv const \text{ and } f(a) = f(b) \right\}.$$

It is clear that the both algebras A_1 and A_2 don't disappear at any point of the compact $K = [a, b]$ and all sets $[x]_{A_1} \cap [x]_{A_2} = \{x\}$ are one-point. Then for $A = A_1 + A_2$ we have $E_R^A(K) = C_R[a, b]$.

Let $f = f_1 + f_2 \in E_R^{A_1}(K) + E_R^{A_2}(K)$, where $f_1 \in E_R^{A_1}(K)$ and $f_1 \in E_R^{A_2}(K)$. By definition of the classes $E_R^{A_1}(K)$ and $E_R^{A_2}(K) : f_1(a) = f_1(c)$, $f_1(d) = f_1(b)$, $f_2(a) = f_2(b)$, and $f_2(c) = f_2(d)$. Hence, $f(a) + f(d) = f_1(a) + f_2(a) + f_1(d) + f_2(d) = f_1(c) + f_2(b) + f_1(b) + +f_2(c) = f(c) + f(b)$. This relation holds for any function $f \in \overline{E_R^{A_1}(K) + E_R^{A_2}(K)}$, i.e. $\overline{E_R^{A_1}(K) + E_R^{A_2}(K)} \neq E_R^A(K) = C_R[a, b]$.

Moreover, the functions

$$f_1(t) = \begin{cases} 0, & \text{at } t \in [a, c], \\ t - c/d - c, & \text{at } t \in [c, d], \\ 1, & \text{at } t \in [d, b], \end{cases}$$

$$f_2(t) = \begin{cases} t - a/c - a, & \text{at } t \in [a, c], \\ 1, & \text{at } t \in [c, d], \\ t - b/d - b, & \text{at } t \in [d, b] \end{cases}$$

belong to the class $\overline{E_R^{A_1}(K) + E_R^{A_2}(K)}$, but for $\varphi = f_1 f_2$ the equality $\varphi(a) + \varphi(d) = = \varphi(c) + \varphi(b)$ does not hold, i.e. $\overline{A_1 + A_2}$ is not algebra.

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