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VANISHING OF THE SECOND VARIATION OF A FUNCTIONAL AS NECESSARY CONDITION OF SINGULAR CONTROLS OPTIMALITY

Abstract

The necessary second order equality type optimality condition of is obtained being the source of series of the known optimality conditions.

In the present paper using the method from [1]considering the general case of generation (an arbitrary rank of Legandre-Clebsh matrix) the necessary optimality condition of equality type in new formulation is obtained.

1. The construction of problem and some necessary informations. Consider the following system

$$x(t) = f(x, u, t), x(t_0) = x_0, \ T \in [t_1, t_1], \tag{1}$$

where $x = (x_1, ..., x_n)'$ is n-vector of phase coordinates, $u = (u_1, ..., u_r)'$ is r -vector of control actions, and the prime means the sign of transposition

Let U be an open set of r-dimensional Euclidean space E^r. Each piecewise continuous function $u(t)$ taking the value from U

$$u(t) \in U, \quad t \in T, \tag{2}$$

we'll call admissible control.

Problem. Minimize the functional

$$S(u) = \varphi(x(t_1)), \tag{3}$$

on trajectory of system (1) generated by the admissible controls (2).

Introduce the following assumptions: 1) the continuous mapping $f(x, u, t)$: $E^n \times E^r \times T \to E^n$ has the continuous partial derivatives by x and u till the second order inclusively and the continuous functional $\varphi(x) : E^n \to (-\infty, +\infty)$ is twice continuously-differentiable; 2) each of admissible control $u(t)$ corresponds the unique absolutely continuous solution of system (1).

The admissible control $u(t)$ being a solution of the problem (1)-(3) we'll call optimal control.

It is known (see for example [2,p53]) that if the optimal control $u(t)$ in problem (1)-(3) exists then it necessarily satisfies the conditions:

$$\delta^1 S(u; \delta u) = -\int_{t_0}^{t_1} H_u'(t) \, \delta u(t) \, dt = 0, \quad \forall \delta u(t) \in \tilde{C}(T, E^r), \tag{4}$$

$$\delta^2 S(u; \delta u) = -\int_{t_0}^{t_1} \left[\delta u'(t) H_{uu}(t) \delta u(t) + 2\delta x'(t) H_{xu}(t) \delta u(t) + \right.$$

$$\left. +\delta x'(t) H_{xx}(t) \delta x(t) \right] dt + \delta x'(t_1) \varphi_{xx}(x(t_1)) \delta x(t_1) \geq 0, \quad \forall \delta u(t) \in \tilde{C}(T, E^r). \tag{5}$$

Here $\delta^1 S(\cdot)$ is the first, and $\delta^2 S(\cdot)$ is the second variations of the functional $S(u)$; $\delta u(t)$ is a variation of optimal control $u(t)$; $\delta x(t)$ is corresponding variation of the trajectory $x(t)$ of system (1) being the solution of the system:

$$\delta \dot{x}(t) = f_x(t)\,\delta x(t) + f_u(t)\,\delta u(t), \quad \delta x(t_0) = 0, \quad t \in T, \tag{6}$$

$$H(\psi, x, u, t) = \psi' f(x, u, t), \quad H_\mu(t) = H_\mu(\psi(t), x(t), u(t), t),$$

$$f_x(t) = f_x(x(t), u(t), t), \quad H_{\mu\nu}(t) = H_{\mu\nu}(\psi(t), x(t), u(t), t), \quad \mu, \nu \in \{x, u\},$$

$\psi(t)$ is a solution of conjugate system

$$\dot{\psi}(t) = -H_x(t), \quad \psi(t_1) = -\varphi_x(x(t_1)),$$

$\tilde{C}(T, E^r)$ is a class of piecewise-continuous vector-functions $g : T \to E^r$.

Form (4), (5) it easily follows the classical necessary optimality conditions (analogies of Euler and Legandre-Clebsh equation)

$$H_u(t) = 0, \quad u' H_{uu}(t) \le 0, \ \forall t \in T, \ \forall u \in E^r. \tag{7}$$

Definition 1. *The admissible control $u(t)$ satisfying (7) we'll call singular (in classical sense) on T, if*

$$rang H_{uu}(t) = r_1 < r, \ t \in T.$$

Let $u(t) = (\upsilon(t), w(t))'$ where $\upsilon(t) = (\upsilon_1(t), ..., \upsilon_{r_0}(t))'$, $w(t) = (w_1(t), ..., w_{r_1}(t))'$, $r_0 + r_1 = r$. Not losing generality [2, p.138] we'll assume that the singularity of control $u(t)$ delivers the vector component $\upsilon \in E^{r_0}$, i.e.

$$H_{\upsilon\upsilon}(t) = 0, \ t \in T. \tag{8}$$

2. The second functional on new variation control (variation of global type). Let $u(t) = (\upsilon(t), w(t))'$ be singular control on T satisfying condition (8).

Consider the following variation of control:

$$\delta u(t') = \begin{cases} 0, \quad t \in [t_0, \theta), \quad 0 \in E^r, \\ (\upsilon, o)', \ t \in [\theta, \theta + \varepsilon) \subset T, \ \upsilon \in E^{r_0}, \ o \in E^{r_1}, \\ \varepsilon \delta \tilde{u}(t), \ t \in [\theta + \varepsilon t_1], \ \delta \tilde{u}(t) \in \tilde{C}(T, E^r), \end{cases} \tag{9}$$

Using Cauchy formula for solution $\delta x(t)$ of system (6) corresponding to variation (9) we can get the representation

$$\delta x(t) = \begin{cases} 0, \quad t \in [-\infty, \theta), \\ \lambda(\theta, t) f'_\upsilon(\theta)\, \upsilon(t - \theta) + o(\varepsilon; t), \ t \in [\theta, \theta + \varepsilon), \\ \varepsilon [\lambda(\theta, t) f_\upsilon(\theta)\, \upsilon + \delta \tilde{x}(t; \theta)] + o(\varepsilon; t), \ t \in [\theta + \varepsilon, t_1], \end{cases} \tag{10}$$

where $\lambda(s, t), \ t_0 \le s < t \le t_1$ is a solution of the system

$$\lambda_t = (s, t) = f_x(t)\, \lambda(s, t), \ \lambda(s, s) = E \quad (E \text{ is a unique } n \times n \text{ matrix});$$

and $\delta \tilde{x}(t, x)$ is determined by the following way:

$$\delta \tilde{x}(t; s) = \int_s^t \lambda(\tau, t) f_u(\tau)\, \delta \tilde{u}(\tau), \ t_0 \le s \le t \le t_1, \tag{11}$$

Subject to (8) and properties $\delta x(t) \sim \varepsilon,\ t \in [t_0, t_1]$ (see (10)) the second variation $\delta^2 S(u;\delta u)$ (see (5)) on variation (9) we can write in the form

$$\delta^2 S(u;\delta u) = -2\int\limits_{\theta}^{\theta+\varepsilon} \delta x'(t) H_{xv}(t) v dt - \int\limits_{\theta+\varepsilon}^{t_1} \delta x'(t) H_{xx}(t)\delta x(t) dt -$$

$$-\varepsilon \int\limits_{\theta+\varepsilon}^{t_1} \left[\delta x'(t) H_{xu}(t)\delta\tilde{u}(t) + \delta\tilde{u}'(t) H_{ux}(t)\delta x(t)\right] dt - \varepsilon^2 \int\limits_{\theta+\varepsilon}^{t_1} \left[\delta\tilde{u}'(t) H_{uu}(t)\delta\tilde{u}(t) dt + \right.$$

$$+\delta x'(t_1)\varphi_{xx}(x(t_1))\delta x(t_1) + o\left(\varepsilon^2\right).$$

Hence by virtue of (10), (11) after the elementary transformations we'll obtain

$$\delta^2 S(u;\delta u) = -\varepsilon^2 \left\{ v'\left[f_v'(\theta) H_{xv}(\theta) + f_v'(\theta)\Psi(\theta) f_v(\theta)\right] v + \right.$$

$$\left. +2v' f_v'(\theta) N(\theta;\delta u) + M(\theta;\delta\tilde{u}) \right\} + o\left(\varepsilon^2\right), \tag{12}$$

where $v \in E^{r_0},\ \delta\tilde{u}(t) \in \tilde{C}(T, E^r),\ \Psi(t),\ t \in T$ is a solution of system [2]:

$$\dot{\Psi}(t) = -f_x'(t)\Psi(t) - \Psi(t) f_x(t) - H_{xx}(t),\ t \in [t_0, t_1),\ \Psi(t_1) = -\varphi_{xx}(x(t_1)), \tag{13}$$

$$N(s;\delta\tilde{u}) = \int\limits_{s}^{t_1} \left[\lambda'(s,t) H_{xx}(t)\delta\tilde{x}(t;s) + \lambda'(s,t) H_{xu}(t)\delta\tilde{u}(t)\right] dt -$$

$$-\lambda'(s,t_1)\varphi_{xx}(x(t_1))\delta\tilde{x}(t_1;s),\ s \in [t_0, t_1),\ \delta\tilde{u}(t) \in \tilde{C}(T, E^r), \tag{14}$$

$$M(s;\delta\tilde{u}) = \int\limits_{s}^{t_1} \left[\delta\tilde{x}(t;s) H_{xx}(t)\delta x\left(t; s + \delta\tilde{x}'(t;s) H_{xu}(t)\delta\tilde{u}(t) + \right)\right.$$

$$\left. +\delta\tilde{u}'(t) H_{ux}(t)\delta\tilde{x}(t;s) + \delta\tilde{u}'(t) H_{uu}(t)\delta\tilde{u}(t)\right] dt -$$

$$-\delta x'(t_1;s)\varphi_{xx}(x(t_1))\delta x(t_1;s),\ s \in T,\ \delta\tilde{u}(t) \in \tilde{C}(T, E^r). \tag{15}$$

The immediate differentiation of (15) subject to (11), (14) and properties $\delta\tilde{x}(s;s) = 0,\ s \in [t_0, t_1]$ allows to show that the relations

$$\frac{dM(s;\delta\tilde{u})}{ds} = -\delta\tilde{u}'(s) f_u'(s) N(s;\delta\tilde{u}) - \left[N'(s;\delta\tilde{u}) f_u(s) - \delta\tilde{u}'(s) H_{uu}(s)\right]\delta\tilde{u}(s),$$

$$M(t_1;\delta\tilde{u}) = 0,\ \delta\tilde{u}(s) \in C(T, E^r),\ s \in T. \tag{16}$$

hold. Further, it is evident that $\delta\tilde{x}(t;t_0) = \delta\tilde{x}(t),\ t \in T$, where $\delta\tilde{x}(t)$ is a solution of system (6) corresponding to $\delta\tilde{u}(t) \in \tilde{C}(T, E^r)$.

Allowing for this property by virtue of (5) and (15) we have

$$\delta^2 S(u;\delta\tilde{u}) = -M(t_0;\delta\tilde{u}),\ \forall \delta\tilde{u}(t) \in \tilde{C}(T, E^r). \tag{17}$$

3. Necessary optimality conditions. On the basis of formula (12) it is easily obtained the following

Theorem 1. *For optimality of singular control $u(t)$ satisfying (8) it is necessary for all $t, \upsilon, \delta\tilde{u}(t)$ $\left(t \in [t_0, t_1),\ \upsilon \in E^{r_0},\ \delta\tilde{u}(t) \in \tilde{C}(T, E^r)\right)$ the inequality*

$$\upsilon' \left[f'_\upsilon(t) H_{\alpha\upsilon}(t) + f'_\upsilon(t)\Psi(t) f_\upsilon(t)\right]\upsilon + 2\upsilon' f'_\upsilon(t) N(t; \delta\tilde{u}(t)) + M(t; \delta\tilde{u}) \le 0, \quad (18)$$

be fulfilled, where $\Psi(\cdot)$, $N(\cdot)$, $M(\cdot)$ are determined by (13), (14), (15) respectively.

Let's prove the following theorem.

Theorem 2. *Let along the singular control $u(t)$ satisfying (8) the condition*

$$R_0(t) + R'_0(t) = 0, \ t \in T, \quad (19)$$

be fulfilled, where $R_0(t) = f'_\upsilon(t) H_{x\upsilon}(t) + f'_\upsilon(t)\Psi(t) f_\upsilon(t), \ t \in T.$

Then for the optimality of control $u(t)$ it is necessary for all $\delta\tilde{\upsilon} \in \tilde{C}(T, E^{r_0})$ the equality

$$\delta^2 S\left(u; (\delta\tilde{\upsilon}(t), o)'\right) = 0$$

be fulfilled.

Proof. From (18) by virtue of (19) we obtain that inequality $2\upsilon' f'_\upsilon(t) N(t; \delta\tilde{u}) \le 0$ was fulfilled for all $t \in T,\ \upsilon \in E^{r_0},\ \delta\tilde{u}(t) \in \tilde{C}(T, E^r)$. Hence subject to (14), (15) it follows that

$$\upsilon' f'_\upsilon(t) N(t; \delta\tilde{u}) = 0, \ \forall t \in T, \ \forall \upsilon \in E^{r_0}, \ \forall \delta\tilde{u}(t) \in \tilde{C}(T, E^r). \quad (20)$$

Then 1) from (20) particularly we have

$$\delta\tilde{\upsilon}(t) f'_\upsilon(t) N\left(t; (\delta\tilde{\upsilon}(t), o)'\right) = 0, \ \forall t \in T, \ \forall \delta\tilde{\upsilon}(t) \in \tilde{C}(T, E^{r_0}),$$

2) by virtue of (8) the equality becames valid

$$(\delta\tilde{\upsilon}(t), o)' H_{uu}(t) (\delta\tilde{\upsilon}(t), o) = 0, \ \forall t \in T, \ \forall \delta\tilde{\upsilon}(t) \in \tilde{C}(T, E^{r_0}).$$

Substituting them in system (16) we arrive at the relation

$$M\left(t; (\delta\tilde{\upsilon}(t), o)'\right) = 0, \forall t \in T, \ \forall \delta\tilde{\upsilon}(t) \in \tilde{C}(T, E^{r_0}). \quad (21)$$

So, equality (21) and properties (17) allows to complete the proof of theorem 2.

Note, that theorem 1 is generalization and strengthening of the result from [2, p.230]. Theorem 2 is a source of series of the known necessary optimality conditions of equality type. Further, one of the main applications of theorem 1 is the investigation of optimality of singular controls on the basis of the third, fourth variation of a functional.

References

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