Farman I. MAMEDOV, Mirfaig M. MIRHEYDARLI

ON SOLVABILITY OF NEUMANN PROBLEM FOR CORDESS TYPE QUASILINEAR ELLIPTIC EQUATIONS

Abstract

In the paper the strong solvability of Neuman problem

$$\sum_{i,j=1}^{n} a_{ij}\left(x,u,u_x\right)\frac{\partial^2 u}{\partial x_i \partial x_j}-\frac{\omega^2 Tra}{n-1}u=b\left(x,u,u_x\right)$$

a.e. $x\in D$*;*

$$\frac{\partial u}{\partial n}=0$$

in Sobolev space $\tilde{W}^{2,2}(D)$*, for some class of quasilinear elliptic equations with parameter* ω*, whose leading coefficents satisfy Cordess condition has been proved.*

§1. **Introduction.**

Let $D\subset E_n$ be a bounded domain of n -dimensional Euclidean space of the points $x=(x_1,x_2,...,x_n)$, $n\geq 2$. The boundary ∂D of the domain D belongs to the class C^2. Consider the second boundary value problem in the domaind:

$$\sum_{i,j=1}^{n} a_{ij}\left(x,u,u_x\right)u_{ij}-\omega^2\frac{T_r a}{n-1}u=b\left(x,u,u_x\right), \tag{1}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial D}=0, \tag{2}$$

where ω is a real number, $T_r a=\sum\limits_{i=1}^{n} a_{ii}\left(x,u,u_x\right)$, $u_x=(u_1,u_2,...,u_n)$, $u_i=\dfrac{\partial u}{\partial x_i}$, $u_{ij}=\dfrac{\partial^2 u}{\partial x_i\partial x_j}$, $i,\ j=1,2,...,\ n;\|a_{ij}(x,z,\theta)\|$ is a real symmetric matrix whose elements are the functions measurable in $x\in D$ for any fixed $z\in E_1,\theta\in E_n$. For almost every $x\in D$ functions $(z,\theta)\to a_{ij}(x,z,\theta)$ and $(z,\theta)\to b(x,z,\theta)$ are continuous in $E_1\times E_n$. Moreover, it is assumed that

$$a)\lambda\left|\xi\right|^2\leq\sum_{i,j=1}^{n} a_{ij}\left(x,z,\theta\right)\xi_i\xi_j\leq\lambda^{-1}\left|\xi\right|^2,\ a.e.\ x\in D,\ z\in E_1,\ \theta\in E_n,\xi\in E_n, \tag{3}$$

$$b)\sup_{x\in D,z\in E_1,\theta\in E_n}\left(\sum_{i,j=1}^{n} a_{ij}^2\left(x,z,\theta\right)\Big/\sum_{i=1}^{n} a_{ij}\left(x,z,\theta\right)\right)^2<\frac{1}{n-1}, \tag{4}$$

$$c)\ |b(x,z,\theta)| \le b_1(x)\left(1+|\theta|^{\nu}\right),\ b_1 \in L_p(D)\ \ for\ p>2,\ \nu \in \left[0, \frac{n(p-2)}{p(n-2)}\right), \quad (5)$$

or $p=2,\ \nu=0$ for any $x \in E_1,\ \theta \in E_n$ a.e. $x \in D$.

Here $\lambda \in (0,1]$ is a constant number. Condition (4) is said to be Cordes condition and is understood in the sense of equivalence of the nondegenerate linear transformation; by means of the non degenerate linear substitution of variables equation (1) is reduced to the form $L'u=b$, whose leading coefficients satisfy the condition (4). We'll assume that the function $b(x,z,\theta)$ is measurable with respect to x in the domain D for any fixed $z \in E_1, \theta \in E_n$. The goal of the paper is to prove the strong solvability of problems (1), (2) in the Sobolev spaces $W_2^2(D)$. Related problems for linear equations were studied in [1].

The questions of classic solvability of Neumann problem for linear equations with smooth coefficients were studied by several authors as M. Schecter, Ya. Lopatinskii, Z.Shapiro, S.Agmon, A.Duglis, L.Nierenberg (see ref. in [1]). Concerning a strong solvability of the mentioned problem for linear equations with continuous coefficients we note O.A. Ladyzhenskaya's papers (see ref. of [1], and [2]). On solvability of Dirichlet problem for elliptic and parabolic equations with discontinuous coefficients satisfying Cordes condition we note the papers by I.Talenti [3], Yu.Alkhutov and I.T. Mamedov [4], [5]. For elliptic equations with leading coefficients from the class VMO the corresponding results were obtained in the papers by C.Vitanza [6], [7] and D.Palagachev [8]. We also note the recent papers by M.Tain [9] and J.Wen [10] devoted to a strong solvability of a mixed boundary value problem for some class of non-linear parabolic equations satisfying the condition close to (3).

Let $1 \le p < \infty$, $W_p^1(D)$ and $W_p^2(D)$ be Banach space of the functions $u(x) \in L_p(D)$ having partial derivatives u_i and u_{ij} in the sense of distributions theory in D belonging to $L_p(D)$ respectively. The norm of spaces $W_p^1(D)$ and $W_p^2(D)$ are given by the forms

$$\|u\|_{W_p^1(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n |u_{ij}|^p\right) dx\right)^{1/p}$$

and

$$\|u\|_{W_p^2(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p\right) dx\right)^{1/p}$$

respectively. For brievity of denotation we shall write the norms of the spaces $W_p^1(D), W_p^2(D)$ and $L_p(D)$ as $\|u\|_{1,p}, \|u\|_{2,p}$ and $\|u\|_p$, respectively. We'll say that the function $u(x) \in W_2^2(D)$ belongs to $\tilde{W}_2^2$, if for any function $\eta(x) \in W_2^1(D)$ it is fulfilled the identity

$$\int_D \eta \Delta u dx = -\int_D \left(\sum_{i=1}^n \eta_i u_i\right) dx, \quad (6)$$

where Δ is a Laplace operator in E_n. It is clear that $\tilde{W}_2^2(D)$ is a space of functions from $\tilde{W}_2^2(D)$ where all the functions $u(x)\in C^\infty\left(\bar{D}\right)$ are dense, for which $\left.\frac{\partial u}{\partial n}\right|_{\partial D}=0$. By $mes_n D$ we'll denote n-dimensional Lebesque measure of the domain D. Everywhere in this paper by $C, C_1, C_2, ...$ we'll denote the constants whose values depend on n and constants in the conditions (1)-(4).

§2. Strong solvability of Neumann problem

For linear equations let $Q=D\times\left(0,\frac{2\pi}{\omega}\right)$, $\tilde{L}$ be an operator $\sum\limits_{i,j=1}^{n}\tilde{\alpha}_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}$ where $\{a_{ij}(x)\}_{i,j=1}^{n}$ is a positive matrix of functions satisfying the conditions (3), (4), $\tilde{\alpha}_{ij}(x)=\frac{(n-1)\,a_{ij}(x)}{T_r a}$; $i,j=1,2,...,n$, $T_r a=\sum\limits_{i=1}^{n}a_{ii}(x)$. Denote $\tilde{\Delta}=\Delta+\frac{\partial^2}{\partial t^2}$ Laplace operator of $(n+1)$ variables. For the function $\tilde{u}: D\times\left(0,\frac{2\pi}{\omega}\right]\to R^1$ we denote $u_{xx}^2=\sum\limits_{i,j=1}^{n}u_{ij}^2$, $u_{xt}^2=\sum\limits_{i=1}^{n}u_{it}^2$ where u_t, u_{it} and u_{tt} mean $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x_i\partial t}$ and $\frac{\partial^2 u}{\partial t^2}$ respectively. Let ν_Q denote a mean value of the function $\nu(x,t)$ on the domain Q:

$$\nu_Q=\frac{1}{mes_{n+1}Q}\int\limits_{Q}\int\nu(x,t)\,dxdt.$$

Let

$$\delta=\sup_{x\in D}\left(\sum_{i,j=1}^{n}\left(\frac{n-1}{T_r a}a_{ij}(x)-\delta_{ij}\right)^2\right)^{1/2}.$$

By condition (4) we get $\delta<1$, in fact,

$$\delta^2=\sup_{x\in D}\left(\sum_{i,j=1}^{n}\frac{(n-1)^2}{(T_r a)^2}a_{ij}^2(x)-2\sum_{i=1}^{n}\frac{(n-1)\,a_{ij}(x)}{T_r a}\delta_{ij}\right)+$$

$$+\sum_{i,j=1}^{n}\delta_{ij}^2<n-1-2(n-1)+n=1, \tag{7}$$

where δ_{ij} is a Kronecker symbol, $\delta_{ij}=\begin{cases}1, & i=j,\\ 1, & i\neq j.\end{cases}$

Auxilary problem 1. Let $\tilde{f}(x,t)\in L_2(Q)$ be an arbitrary function. Find such a function $\nu(x,t)\in\tilde{W}_2^2(Q)$, $\nu_Q=0$ that for any $\varphi(x,t)\in\tilde{W}_2^2(Q)$, $\varphi_Q=0$ it holds the integral identity

$$\int\limits_{Q}\int\left(\tilde{L}\nu+\nu_{tt}\right)\tilde{\Delta}\varphi\,dxdt=\int\limits_{Q}\int\tilde{f}(x,t)\,\tilde{\Delta}\varphi\,dxdt. \tag{8}$$

Proposition 1. Let the conditions (3)-(4) be fulfilled for the coefficients $\{a_{ij}(x)\}$; $i,j=1,2,...,n$. Then for any function $\tilde{f}(x,t)\in L_2(Q)$ the problem 1 has a unique solution $\nu(x,t)\in\tilde{W}_2^2(Q)$, $\nu_Q=0$ and for the solution the estimation

$$\|\nu\|_{2,2}\le C\left\|\tilde{f}\right\|_2, \tag{9}$$

is valid, where

$$C=C_1\sqrt{1+C_0\left(\frac{mes_n D}{\omega}\right)^{\frac{2}{n+1}}+C_0^2\left(\frac{mes_n D}{\omega}\right)^{\frac{4}{n+1}}}. \tag{10}$$

$C_0,C_1>0$ depend on n,λ,δ.

Proof. Consider the bilinear form

$$B(\nu,\varphi)=\int\int_Q\left(\tilde{L}\nu+\nu_{tt}\right)\tilde{\Delta}\varphi dxdt,$$

in $\tilde{W}_2^2(Q)\times\tilde{W}_2^2(Q)$, where $\forall\varphi\in\tilde{W}_2^2(Q),\varphi_Q=0,\nu_Q=0$.

Problem 1 will be written in the form

$$B(\nu,\varphi)=\int\int_Q\tilde{f}(x,t)\tilde{\Delta}\varphi dxdt. \tag{11}$$

$\forall\varphi\in\tilde{W}_2^2(Q),\ \nu\in\tilde{W}_2^2,\ \nu_Q=0,\ \varphi_Q=0.$

Apply the Lax-Millgram principle to the solvability of problem 1.

The form $B(\nu,\varphi)$ is continuous. Really,

$$|B(\nu,\varphi)|\le\left\|\tilde{\Delta}\varphi\right\|_2\left\|\tilde{L}\nu+\nu_{tt}\right\|_2\le$$

$$\le\left\|\tilde{\Delta}\varphi\right\|_2\left(\int\int_Q\left[2\left(\left(\tilde{L}\nu\right)^2+\nu_{tt}^2\right)dxdt\right]dxdt\right)^{1/2}. \tag{12}$$

By Hőlder inequality and (4)

$$\left(\tilde{L}\nu\right)^2\le\left(\sum_{i,j=1}^n\frac{a_{ij}^2(n-1)^2}{(T_ra)^2}\right)\sum_{i,j=1}^n\nu_{ij}^2\le(n-1)\sum_{i,j=1}^n\nu_{ij}^2=(n-1)\nu_{xx}^2.$$

By this inequality we get from (12)

$$|B(\nu,\varphi)|\le\sqrt{2(n-1)}\left\|\tilde{\Delta}\varphi\right\|_2\|\nu\|_{2.2}\le\sqrt{2(n^2-1)}\|\varphi\|_{2.2}\|\nu\|_{2.2}. \tag{13}$$

The coerciveness of the form $B(\nu,\varphi)$:

$$B(\nu,\nu)\int\int_Q\left[\tilde{L}\nu^2+\nu_{tt}\right]\tilde{\Delta}\nu dxdt=$$

$$= \int\int_Q \left[\left(\sum_{i,j=1}^{n} \frac{n-1}{T_r a} a_{ij}(x) - \delta_{ij} \right) \nu_{ij} \tilde{\Delta}\nu \right] \tilde{\Delta}\nu dxdt \geq \tag{14}$$

$$\geq \int\int_Q \left(\tilde{\Delta}\nu\right)^2 dxdt - \int\int_Q \left(\sum_{i,j=1}^{n} \left(\frac{n-1}{T_r a} a_{ij}(x) - \delta_{ij} \right)^2 \right)^{1/2} \left(\sum_{i,j=1}^{n} \nu_{ij}^2 \right)^{1/2} \left|\tilde{\Delta}\nu\right| dxdt.$$

By condition (7) we get from (14)

$$B(\nu,\nu) \geq \int\int_Q \left(\tilde{\Delta}\nu\right)^2 dxdt - \delta \left(\int\int_Q \left(\tilde{\Delta}\nu\right)^2 dxdt \right)^{1/2} \left(\int\int_Q \nu_{xx}^2 dxdt \right)^{1/2}. \tag{15}$$

Now use the estimation

$$\int\int_Q \left[\nu_{xx}^2 + 2\nu_{xt}^2 + \nu_{tt}^2\right] dxdt \leq \int\int_Q \left(\tilde{\Delta}\nu\right)^2 dxdt \tag{16}$$

for the functions $\nu(x,t) \in \tilde{W}_2^2(Q)$ proved in the paper [1]. Then we'll get from (15), (16)

$$B(\nu,\nu) \geq (1-\delta) \int\int_Q \left(\tilde{\Delta}\nu\right)^2 dxdt \geq$$

$$\geq (1-\delta) \int\int_Q \left[\nu_{xx}^2 + \left(\frac{\partial^2 \nu}{\partial t^2} \right)^2 + 2\nu_{xt}^2 \right] dxdt \geq \frac{1-\delta}{C} \|\nu\|_{2.2}^2 . \tag{16'}$$

In the last inequality we used the fact that for the functions $\nu(x,t) \in \tilde{W}_2^2(Q)$, $\nu_Q = 0$ it is valid the estimation

$$\|\nu\|_{2.2} \leq C \int\int_Q \left(\nu_{xx}^2 + 2\nu_{xt}^2 + \nu_{tt}^2\right) dxdt,$$

where $C > 0$ is a constant from (10). Really, integrating the identity

$$\tilde{\Delta}\nu = 2\nu\tilde{\Delta}\nu + 2\nu_x^2 + 2\left(\frac{\partial \nu}{\partial t}\right)^2$$

on Q, allowing for $\nu \in \tilde{W}_2^2(Q)$ we get

$$\int\int_Q \left[\nu\tilde{\Delta}\nu + \nu_x^2 + \nu_t^2\right] dxdt = 0.$$

Whence by Hőlder inequality

$$\int\int_Q \left[\nu_x^2 + \nu_t^2\right] dxdt \leq \left(\int\int_Q \nu^2(x,t) dxdt \right)^{1/2} \left(\int\int_Q \left(\tilde{\Delta}\nu\right)^2 dxdt \right)^{1/2}. \tag{17}$$

By Poincare inequality (see [11, theorem 4.2]) and that $\nu_Q = 0$ we have

$$\left(\int\int\limits_Q \nu^2(x,t)\,dxdt\right)^{1/2} = \left(\int\int\limits_Q (\nu-\nu_Q)^2\,dxdt\right)^{1/2} \leq$$

$$\leq \left(\left(\int\int\limits_Q |\nu-\nu_Q|^{\frac{2(n+1)}{n-1}}\,dxdt\right)^{\frac{n-1}{2(n+1)}} mes_{n+1}Q\right)^{\frac{1}{n+1}} \leq$$

$$\leq C\left(\int\int\limits_Q \left[\nu_x^2+\left(\frac{\partial\nu}{\partial t}\right)^2\right]dxdt\right)^{1/2}\left(mes_n D\frac{2\pi}{\omega}\right)^{\frac{1}{n+1}}.$$

$$\leq C\left(\int\int\limits_Q \left[\nu_x^2+\left(\frac{\partial\nu}{\partial t}\right)^2\right]dxdt\right)^{1/2}\left(mes_n D\frac{2\pi}{\omega}\right)^{\frac{1}{m+1}}, \tag{18}$$

the constant $C_0 > 0$ depends only on n. By (18) we get from (17)

$$\left(\int\int\limits_Q \left[\nu_x^2+\left(\frac{\partial\nu}{\partial t}\right)^2\right]dxdt\right)^{1/2} \leq$$

$$\leq C_0\left(mes_n D\frac{2\pi}{\omega}\right)^{\frac{1}{n+1}}\left(\int\int\limits_Q \left(\tilde{\Delta}\nu\right)^2 dxdt\right)^{1/2}. \tag{19}$$

By (18) it follows

$$\left(\int\int\limits_Q \nu_x^2\,dxdt\right)^{1/2} \leq C_0^2\left(mes_n D\frac{2\pi}{\omega}\right)^{\frac{2}{n+1}}\left(\int\int\limits_Q \left(\tilde{\Delta}\nu\right)^2 dxdt\right)^{1/2}. \tag{20}$$

We get from (16), (19) and (20)

$$\|\nu\|_{2.2} \leq C\left\|\tilde{\Delta}\nu\right\|_2, \tag{20'}$$

where

$C_0 > 0$ depends only on n, Q.E.D.

By $\tilde{f}(x,t) \in L_2(Q)$ the functional

$$\int\int\limits_Q \tilde{f}(x,t)\,\tilde{\Delta}\varphi\,dxdt$$

belongs to $\left(\tilde{W}_2^2\right)^*$:

$$\left|\int\int_Q \tilde{f}(x,t)\,\tilde{\Delta}\varphi dxdt\right| \le$$

$$\le \left(\int\int_Q \left|\tilde{f}(x,t)\right|^2 dxdt\right)^{1/2} \left(\int\int_Q \left|\tilde{\Delta}\varphi\right|^2 dxdt\right)^{1/2} \le$$

$$\le \sqrt{n+1}\left\|\tilde{f}\right\|_2 \|\varphi\|_{2.2}\,. \tag{21}$$

Now, after that we established the continuity and coerciveness of the form $B(\nu,\varphi)$ we can apply the Lax-Millgram principle to the solvability of problem 1. Then for any $\tilde{f}(x,t)\in L_2(Q)$ we'll get the existence of a unique function $\nu(x,t)\in \tilde{W}_2^2(Q)$, $\nu_Q=0$ satisfying the integral identity (11) for any $\forall\varphi\in\tilde{W}_2^2(Q)$, $\varphi_Q=0$. By (16') and (21) for $\varphi=\nu$ from (11) we get the estimation

$$\|\nu\|_{2.2}\le \frac{C}{1-\delta}\sqrt{2(n+1)}\left\|\tilde{f}\right\|_2,$$

where $C>0$ is the same that in (12). Proposition 1 is proved.

Let the operator

$$L'=\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}.$$

Consider the problem

$$L'u-\omega^2\frac{T_r a}{n-1}u=f(x)\,,x\in D, \tag{22}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial D}=0. \tag{23}$$

Definition. *Let $f(x)\in L_2(D)$. The function $u(x)\in\tilde{W}_2^2(D)$ satisfying equation (22) a.e. $x\in D$ is said to be the solution of problem (22), (23).*

Theorem 1. *Let the system of function $\{a_{ij}(x)\};ij=1,2,...,n$ satisfy the conditions (3) and (4). Then for any $f(x)\in L_2(D)\,,\omega\neq 0$ problem (22), (23) has a unique solution from the space $\tilde{W}_2^2(Q)$. And for the solution it is valid the estimation*

$$\|u\|_{2.2}\le C\,\|f\|_2\,, \tag{24}$$

where

$$C=C_1\sqrt{1+C_0\left(\frac{mes_n D}{\omega}\right)^{\frac{2}{n+1}}+C_0^2\left(\frac{mes_n D}{\omega}\right)^{\frac{4}{n+1}}}, \tag{25}$$

$C_0,C_1>0$ are the constants dependent on δ,n,λ.

Proof. Let the denotation given before proposition 1 be fulfilled. Let ν be the solution of problem 1 for $\tilde{f} = f(x)\dfrac{n-1}{T_r a}\cos\omega t$ from the space $\tilde{W}_2^2(Q)$, $\nu_Q = 0$. Then for any $\varphi \in \tilde{W}_2^2(Q)$, $\varphi_Q = 0$ we'll have

$$\int\int_Q \left[\tilde{L}\nu + \nu_{tt} - \tilde{f}\right]\tilde{\Delta}\varphi dxdt = 0. \tag{25'}$$

Assume that Ψ is an arbitrary function from $L_2(Q)$ provided $\Psi_Q = 0$. Choose the function φ in identity (25') as the solution of the following Neumann problem

$$\left.\begin{array}{ll} \tilde{\Delta}\varphi = \Psi, & (x,t) \in Q, \\ \left.\dfrac{\partial\varphi}{\partial n}\right|_{\partial Q} = 0, & \varphi_Q = 0. \end{array}\right\}. \tag{26}$$

Problem (26) is uniquely solvable in the class $\tilde{W}_2^2(Q)$ in view of the condition $\Psi_Q = 0$ on the function Ψ and $\partial D \subset C^2$ on the boundary of the domain D (see [12]). Then it follows from (25')

$$\int\int_Q \left(\tilde{L}\nu + \nu_{tt} - \tilde{f}\right)\Psi(x,t) = 0. \tag{27}$$

If $\Psi \in L_2(Q)$ is an arbitrary function for which $\Psi_Q \neq 0$ then assuming $\Psi - \Psi_Q$ instead of Ψ we get from (27)

$$\int\int_Q \left(\tilde{L}\nu + \nu_{tt} - \tilde{f}\right)\Psi dxdt = \Psi_Q(x,t)\int\int_Q \left(\tilde{L}\nu + \nu_{tt} - \tilde{f}\right)dxdt.$$

Whence

$$\int\int_Q \left(\tilde{L}\nu + \nu_{tt} - \tilde{f} - C_1\right)\Psi dxdt = 0, \tag{28}$$

where

$$C_1 = \frac{1}{mes_{n+1}Q}\int\int_Q \left(\tilde{L}\nu + \nu_{tt} - \tilde{f} - C_1\right)dxdt.$$

From (28) by the arbitrariness of the function $\Psi \in L_2(Q)$ we get

$$\tilde{L}\tilde{u} + \tilde{u}_{tt} - \tilde{f} \equiv C_1. \tag{29}$$

Multiply equation (29) by $\cos\omega t$ and integrate with respect to $\left(0, \dfrac{2\pi}{\omega}\right)$:

$$\int_0^{2\pi/\omega} \tilde{L}(\nu\cos\omega t)\,dt + \int_0^{2\pi/\omega} \nu_{tt}\cos\omega t dt - \int_0^{2\pi/\omega} \tilde{f}\cos\omega t dt = 0,$$

integrating by parts in the second summand we get

$$\tilde{L}z-\omega^2 z=\frac{\omega}{\pi}\int\limits_0^{2\pi/\omega}\tilde{f}\cos\omega t dt, \tag{30}$$

a.e. $x\in D$, where $z(x)=\frac{\omega}{\pi}\int\limits_0^{2\pi/\omega}\nu(x,t)\cos\omega t dt.$

Allowing for the form $\tilde{f}(x,t)=f(x)\frac{(n-1)}{T_r a}\cos\omega t$ of the function $\tilde{f}(x,t)$ we get from (28)

$$\tilde{L}z-\omega^2 z=\frac{n-1}{T_r a}f(x)\quad \text{a.e. } x\in D. \tag{31}$$

Multiplying by $\frac{T_r a}{n-1}$ equation (31) we get that the function $z(x)$ is the solution of equation (22). Obviously, $z(x)\in\tilde{W}_2^2(Q)$, thereby we prove the existence of the solution of problem (22), (23). Using the representation $z(x)=\frac{\omega}{\pi}\int\limits_0^{2\pi/\omega}\nu(x,t)\cos\omega t dt$ and a priori estimation (9) for $\nu(x,t)$ we show (24). Obviously,

$$z_{ij}(x)=\frac{\omega}{\pi}\int\limits_0^{2\pi/\omega}\nu_{ij}(x,t)\cos\omega t dt \text{ a.e } x\in D.$$

Integrating this identity with respect to D by means of Minkovskii inequality we get i.e.

$$\|z_{ij}\|_2=\frac{\omega}{\pi}\int\limits_0^{2\pi/\omega}\|\cos\omega t\|\,\|\nu_{ij}(.,t)\|_2\,dt\le$$

$$\le\frac{\omega}{\pi}\left(\int\limits_0^{2\pi/\omega}\cos^2\omega t dt\right)^{1/2}\left(\int\limits_0^{2\pi/\omega}\|\nu_{ij}(.,t)\|_2^2\,dt\right)^{1/2}=$$

$$=\frac{\omega}{\pi}\sqrt{\frac{\pi}{\omega}}\|\nu_{ij}\|_2\le\sqrt{\frac{\omega}{\pi}}C\left\|\tilde{f}\right\|_2=$$

$$=\sqrt{\frac{\omega}{\pi}}C\left(\int\limits_0^{2\pi/\omega}\int\limits_Q f^2(x)\left(\frac{n-1}{T_r a}\right)^2\cos^2\omega t dx dt\right)^{1/2}=$$

$$=C\left(\int\limits_D f^2(x)\,dx\right)^{1/2}\frac{n-1}{n\lambda},$$

i.e. $\|z_{ij}\|_2\le C\|f\|_2$. The similar estimations are valid also for $z_i,z;i,j=1,2,...,n$.

Then

$$\|z\|_{2.2}\le C\|f\|_2, \tag{32}$$

where $C>0$ is dependent only on n,λ,δ. Estimation (32) yields the uniqueness of problem (22), (23) for any $\omega\neq 0, f(x)\in L_2(D)$.

Passage to the limit. In estimation (32) the constant C is of the form

$$C=\frac{C_1}{1-\delta}\sqrt{1+C_0^4\left(mes_nD\frac{2\pi}{\omega}\right)^{\frac{4}{n+1}}+C_0^2\left(mes_nD\frac{2\pi}{\omega}\right)^{\frac{4}{n+1}}},$$

whence it follows that it depends on $L_2(D)$, moreover $C\to\infty$ as $\omega\to 0$. Below we'll refine some constants for the $L_2(D)$ norm of functions u_x and u_{xx}. By estimations [1, theorem 2] for $u\in\tilde{W}_2^2(D)$ we have

$$\int\limits_D u_{xx}^2dx\le\int\limits_D(\Delta u)^2dx\le C\int\limits_D Lu\Delta udx. \tag{33}$$

Let $u(x)\in\tilde{W}_2^2(D)$ be the solution of the equation

$$\tilde{L}u-\omega^2u=fu,$$

then

$$\tilde{L}u\Delta u-\omega^2u\Delta u=f\Delta u,$$

$$\int\limits_D\tilde{L}u\Delta udx+\omega^2\int\limits_D u_x^2dx=\left(\int\limits_D f^2dx\right)^{1/2}\int\limits_D(\Delta u)^2dx,$$

$$\int\limits_D\tilde{L}u\Delta udx\le\left(\int\limits_D f^2dx\right)^{1/2}\left(\int\limits_D(\Delta u)^2dx\right)^{1/2}, \tag{34}$$

whence

$$\int\limits_D u_{xx}^2dx\le C\left(\int\limits_D f^2dx\right)^{1/2}\left(\int\limits_D u_{xx}^2dx\right)^{1/2},$$

$$\int\limits_D u_{xx}^2dx\le C\int\limits_D f^2dx, \tag{35}$$

where $C>0$ is independent on ω.

On the other hand, by $u\in\tilde{W}_2^2(D)$ we have

$$\int\limits_D u_{xx}^2dx-\int\limits_D u\Delta udx=\int\limits_D\Delta u(u_D-u)dx-u_D\int\limits_D\Delta udx\le\int\limits_D|u-u_D||\Delta u|dx.$$

By Poincare inequality hence and (35) we have

$$\int\limits_D u_x^2dx\le\left(\int\limits_D|u-u_D|^2dx\right)^{1/2}\left(\int\limits_D(\Delta u)^2dx\right)^{1/2}\le$$

$$\leq C\left(\int_D u_x^2 dx\right)^{1/2}\left(\int_D f^2 dx\right)^{1/2},$$

$$\int_D u_x^2 dx \leq C\int_D f^2 dx, \tag{36}$$

here again $C>0$ is independent on ω. We get from (35), (36)

$$\int_D \left(u_x^2+u_{xx}^2\right)dx \leq C\int_D f^2 dx. \tag{37}$$

Denote by $\{u^{\omega}(x)\}$ a family of solutions of the problem

$$\tilde{L}u^{\omega}-\omega^2 u^{\omega}=f, \tag{38}$$

By estimation (37) a family of functions

$$\{g^{\omega}(x)=u^{\omega}(x)-u_D^{\omega},\ 0<\omega<\omega_0\}$$

have a uniformly bounded norm,

$$\|g^{\omega}\|_{2.2}\leq C\|f\|_2, \tag{39}$$

where $C>0$ is independent on ω, since, $g_D^{\omega}=0$ by Poincare inequality and (37) we have

$$\|g^{\omega}\|_2=\|g^{\omega}-g_D^{\omega}\|_2\leq C\|f\|_2.$$

Then from $\{g^{\omega}\}$ we can select a subsequence weakly convergent in $\tilde{W}_2^2(D)$, i.e

$$g^{\omega_k}\to g \text{ in } \tilde{W}_2^2(D) \text{ for } \omega_k\to 0.$$

From (38) we get $\forall\varphi\in\tilde{W}_2^2(D)$

$$\int_D \tilde{L}u^{\omega_k}\Delta\varphi dx-\omega_k^2\int_D u^{\omega_k}\Delta\varphi dx=\int_D f\Delta\varphi dx,$$

integrating by parts

$$\int_D \tilde{L}u^{\omega_k}\Delta\varphi dx+\omega_k^2\int_D \nabla u^{\omega_k}\nabla\varphi dx=\int_D f\Delta\varphi dx, \tag{40}$$

whence

$$\int_D \tilde{L}g^{\omega_k}\Delta\varphi dx+\omega_k^2\int_D \nabla g^{\omega_k}\nabla\varphi dx=\int_D f\Delta\varphi dx,$$

passing to the limit by using the weak convergence

$$g^{\omega_k}\to g$$

in $\tilde{W}_2^2(Q)$ and uniform boundedness of integrals

$$\left|\int\limits_D \nabla g^{\omega_k} \nabla \varphi dx\right| \leq C \left\|\nabla g^{\omega_k}\right\|_2 \left\|\nabla \varphi\right\|_2 \leq C_1 \left\|f\right\|_2 \left\|\nabla \varphi\right\|_2$$

with respect to ω_k, we get

$$\int\limits_D \left(\tilde{L}g - f\right) \Delta\varphi dx = 0, \quad \forall \varphi \in \tilde{W}_2^2(D) \tag{41}$$

whence as above,

$$\tilde{L}g - f \equiv const \;\text{ a.e. }\; x \in D.$$

So, we proved that for any $f(x) \in L_2(D)$ the solution of a classic Neumann problem (22) ($\omega = 0$) exists in the following sense: there will be found such a constant C, the function $g \in \tilde{W}_2^2(D), g_D = 0$ that

$$Lg = f + C \;\text{ a.e. }\; x \in D.$$

Estimation (39) yields the estimation

$$\|g\|_{2,2} \leq C \|f\|_2 ,$$

for the function g, where $C > 0$ depends on λ, n, δ. It follows from the mentioned estimation that the function g is defined uniquely according to the function $f(x) \in L_2(D)$.

We proved the following theorem:

Theorem 2. *Let conditions (3) and (4) be fulfilled with respect to the functions $\{a_{ij}(x)\}$, $i, j = 1, 2, ..., n$. Then for any $f(x) \in L_2(D)$ the Neumann problem (22), (23) for $\omega = 0$ has a unique solution in the following sense: there will be found such a constant C and the function $u \in \tilde{W}_2^2(D)$, $u_D = 0$ that*

$$Lu = f + C \;\textit{a.e.}\; x \in D.$$

Moreover, for the function u it is valid the estimation

$$\|u\|_{2,2} \leq C_1 \|f\|_2 ,$$

where the constant $C_1 > 0$ is dependent on $\lambda, n, \delta, \; mes_n D$.

Remark 1. Let $\{u^s(x)\}, s = 1, 2, ...$ be a sequence of solutions of the problem

$$\begin{cases} L_s u^s - \omega^2 \dfrac{T_r a}{n-1} = f^s(x) \quad \text{for a.e. } x \in D, \\ \left.\dfrac{\partial u^s}{\partial n}\right|_{\partial D} = 0, \end{cases} \tag{42}$$

where $L_s = \sum_{i,j=1}^{n} a_{ij}^s(x) \frac{\partial^2}{\partial x_i \partial x_j} - \omega$, moreover $\{a_{ij}^s(x)\}, ij = 1, 2, ...$ satisfy conditions (3), (4) uniformly on s. Then it is easy to see from the proof of theorem 1 that, it holds the estimation

$$\|u^s\|_{2.2} \leq C \|f_s\|_2 , \tag{43}$$

where $C > 0$ depends only on λ, n, δ.

§3. Strong solvability of Neuman problem for nonlinear equations.

Theorem 3. *Let $\omega \neq 0$, conditions (3)-(5) be fulfilled for the data of problem (1)-(2). Let $\nu = 1$ $(p = n)$ and the measure of the domain D be sufficiently small or $\nu \neq 1$ $(p \neq n)$ and the measure of the domain D be arbitrary. Then any there exists the solution of problem (1)-(2) from the space $\tilde{W}_2^2(D)$.*

Proof. Apply Schauder principle on the existence of a fixed point at continuous mapping of a convex compact onto its part in a Banach space. By A we denote the set

$$\{u(x) : u(x) \in \tilde{W}_2^2(D) \cap W_q^1(D),\ \|u\|_{2,2} \leq N\}.$$

We'll select the numbers N, $1 \leq q < \frac{2n}{n-2}$ later. By a compact embedding theorem $W_2^2(D) \subset W_q^1$ for $1 \leq q < \frac{2n}{n-2}$ the set A is compact. The set A is also convex, really for $u_1, u_2 \in A, \lambda \in (0, 1]$, $u = \lambda u_1 + (1 - \lambda) u_2$ we have

$$\|u\|_{2.2} \leq \lambda \|u_1\|_{2,2} + (1 - \lambda) \|u_2\|_{2,2} \leq \lambda K + (1 - \lambda) K = K, \quad \text{i.e. } u \in A.$$

Let $g \in A$, consider the subsidiary problem

$$\tilde{L}_g u - \frac{\omega^2 T_r a}{n-1} u = \tilde{b}(x), \text{ a.e. } x \in D, \tag{44}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0, \tag{45}$$

where $\tilde{L}_g = \sum_{i,j} \tilde{a}_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$, $\tilde{a}_{ij}(x) = a_{ij}(x, g, g_x)$, $\tilde{b}(x) = b(x, g, g_x)$, $i, j = 1, 2, ...n$. By means of conditions (3) and (4) we find

$$\left\|\tilde{b}\right\|_2 = \left(\int_D b^2(x, g, g_x)\, dx\right)^{1/2} \leq \left(\int_D |(1 + |g_x|^\nu)\, b_1(x)|^2\, dx\right)^{1/2} \leq$$

$$\leq \|b_1\|_p \|1 + |g_x|^\nu\|_{p/(p-2)}^{1/2} \leq \left(\|1\|_{p/(p-2)}^{1/2} + \left\||g_x|^{2\nu}\right\|_{p/(p-2)}^{1/2}\right) \|b_1\|_p ,$$

whence allowing $\|1\|_{p/(p-2)} = (mes_n D)^{(p-2)/p}$ and $0 \le \nu \le \dfrac{n(p-2)}{p(n-2)}$ for $p \ge 2$ by compact imbedding theorem $W_2^2(D) \subset W_q^1(D)$ $\left(1 \le q < \dfrac{2n}{n-2}\right)$ we have

$$\left\| |g_x|^{2\nu} \right\|_{p/(p-2)}^{1/2} \le C(D) \|g\|_{2,2}^{\nu} \le C(D) N^{\nu},$$

where $C(D) \to 0$ when $mes_n D \to 0,\ p > 2$. As a result we get the estimation

$$\left\|\tilde{b}\right\|_2 \le C(D) N^{\nu} \text{ for } p \ge 2. \tag{46}$$

The belongness $\tilde{b} \in L_2(D)$ follows from (46). The problem (44), (45) is uniquely solvable in $\tilde{W}_2^2(D)$ and by remark 1 for its solution estimation (43) is valid. Consider the mapping $T : g \in A \to u \in \tilde{W}_2^2(D)$ is the solution of problem (44), (45). Show that the operator T transfers the function $g \in A$ to the function $u \in A$. By (46) and theorem 1 (and remark 1) we get

$$\|u\|_{2,2} \le C \left\|\tilde{b}\right\|_2 \le C(D) N^{\nu}.$$

If $b_1 \in L_p(D), p > n$, then $\nu > 1$. If we select $0 < N < 1$ sufficiently small we can obtain

$$C \|b\|_p C(D) N^{\nu} \le N, \tag{47}$$

here the measure of the domain D is arbitrary. If $b_1 \in L_p, 1 \le p < n$, then $0 \le \nu < 1$. We must select $N > 1$ sufficiently large in order that condition (47) be fulfilled; the measure of the domain D is arbitrary. We should consider the case $p = n$, i.e $\nu = 1$. Then we are to consider the domain D with sufficiently small Lebesque measure in order (46) be hold.

We got $\|Tg\|_{2.2} \le N$ i.e. $T : A \to A$ is established.

Now show the continuity of the mapping T. Let $\{g^s\}$, $s = 1, 2, ...$ be the sequence of functions convergent on the norm $W_q^1(D)$ to the function g_0. Show that sequence $\{Tg_s\}$ will converge to $u_0 = Tg_0$. Obviously,

$$L_s\left(u^s - u^0\right) = L_s u^s - L_s u^0 = (L_0 - Ls) u^0 + L_s u^s - L_0 u^0 =$$

$$= b\left(x, g^s, g_x^s\right) - b\left(x, g^0, g_x^0\right) + \sum_{i,j=1}^{n} \left[a_{ij}\left(x, g^s, g_x^s\right) - a_{ij}\left(x, g^0, g_x^0\right)\right] u_{ij}^0 +$$

$$+ \frac{T_r a^s - Tra^0}{n-1} u^0 =: F^s(x),\ x \in D,$$

where

$$L_s = \sum_{i,j=1} a_{ij}\left(x, g^s, g_x^s\right) \frac{\partial^2}{\partial x_i \partial x_j} - \omega^2 \frac{T_r a^s}{n-1},$$

$$T_r a^s = \sum_{i,j=1}^{n} a_{ii}^s\left(x, g^s, g_x^s\right), s = 0, 1, 2, ...$$

On the basis of theorem 1 and remark 1 to theorem 1 we have

$$\left\|u^s - u^0\right\|_{2,2} \leq C \left\|F^s\right\|_2 \leq$$

$$\leq \left[C \left\|\tilde{b}_s - \tilde{b}_0\right\|_2 + \sum_{i,j=1}^{n} \left\| \left[a_{ij}\left(x, g^s, g_x^s\right) - a_{ij}\left(x, g^0, g_x^0\right) \right] u_{ij}^0 \right\|_2 + \right.$$

$$\left. + \left\| \frac{T_r a^s - Tra^0}{n-1} u^0 \right\|_2 \right], \qquad (48)$$

where $\tilde{b} = b\left(x, g^s, g_x^s\right), s = 0, 1, 2, ...,\ C > 0$ is independent on $s, g^s, u^s(x)$. By

$$\left\|g^s - g^0\right\|_{1,q} \text{ as } s \to \infty.$$

There exists a subsequence of the sequence $\{g^s\}$, that we'll denote as the sequence itself $\{g^s\}$ and for which

$$g^s \to g^0,\ g_x^s \to g_x^0 \text{ a.e. } x \in D.$$

Then

$$\left\|\tilde{b}_s - \tilde{b}_0\right\|_2 \to 0 \text{ as } s \to \infty, \qquad (49)$$

by Vitali theorem. Really, by the continuity of functions $b(x, z, \theta)$ on $(z, \theta) \in E_1 \times E_1$ and that $g^s \to g^0,\ g_x^s \to g_x^0$ a.e. $x \in D$ we have $\eta_s = b_s - b_0 \to 0$ as $s \to \infty$ a.e. $x \in D$. On the other hand, the functions $\{\eta_s^2\}$ have absolutely equicontinuous integrals, i.e. for any compact $e \subset D$ on the basis of condition (5) we have

$$\int_e \eta_s^2 dx \leq C \int_e b_1^2(x) \left(1 + |g_x^s|^\nu\right)^2 dx.$$

Now, if $p = 2$ i.e. $\nu = 0$ then we have.

$$\int_e \eta_s^2 dx \leq C \int_e b_1^2(x)\, dx. \qquad (50)$$

If $p > 2$ i.e. $\nu > 0$ by the Hőlder inequality we get

$$\int_e \eta_s^2 dx \leq C_1 \left(\int_e b_1^p(x)\, dx \right)^p (mes_n e)^{\frac{p-2}{p}} +$$

$$+ C_{21} \left(\int_e b_1^p(x)\, dx \right)^{\frac{2}{p}} \left(\int_e |g_x^s|^{\frac{2\nu p}{p-2}} dx \right)^{\frac{p-2}{p}}. \qquad (51)$$

Now select summability exponent q of the space $W_q^1(D)$, so that $\dfrac{2\nu p}{p-2} \leq q$. This is possible by the fact that $0 \leq \nu < \dfrac{n(p-2)}{p(n-2)}$. From the convergence in D

$$\left\|g^s - g^0\right\|_{1,q} \text{ as } s \to \infty.$$

we'll get

$$\|g^s\|_{1,q} \le \|g^s\|_{1,q} + \varepsilon, \tag{52}$$

where $\varepsilon > 0$ is an arbitrary number, the norms are calculated by the set $e \subset D$. Then

$$\int_e |g_x^s|^q \, dx \le \int_e |g_x^0|^q \, dx + \varepsilon \tag{53}$$

at sufficiently large s independent of e. By the arbitrariness of ε we get from estimations (51), (53)

$$\int_e \eta_s^2 (x) \, dx \to 0$$

uniformly in s for $mes_n e \to 0$.

Now after that we established the absolute equicontinuity of integrals

$$\int_e \eta_s^2 (x) \, dx$$

and that $\eta_s (x) \to 0$ a.e. $x \in D$ we can pass to the limit under the sign of integral

$$\lim_{s \to \infty} \int_D \eta_s^2 (x) \, dx = 0. \tag{54}$$

Now show

$$q_s^{ij} = \left\| \left[a_{ij} (x, g^s, g_x^s) - a_{ij} \left(x, g^0, g_x^0\right) \right] u_{ij}^0 \right\|_2 \to 0 \text{ as } s \to \infty; i, j = 1, 2, ..., n.$$

This statement follows from the Lebesque theorem:

$$a_{ij} (x, g^s, g_x^s) \to a_{ij} \left(x, g^0, g_x^0\right) \text{ a.e. } x \in D \text{ as } s \to \infty,$$

by the convergence a.e. $x \in D \quad g^s \to g^0, \; g_x^s \to g_x^0$ and continuity of the functions $a_{ij} (x, z, \theta)$ by the arguments $(z, \theta) \in E_1 \times E_n; \; i, j = 1, 2, ..., n$. By condition (3)

$$\left| \left[a_{ij} (x, g^s, g_x^s) - a_{ij} \left(x, g^0, g_x^0\right) \right] u_{ij}^0 \right|^2 \le \frac{2}{\lambda^2} \left| u_{ij}^0 (x) \right|^2 .$$

The right hand side is the function integrable in D. Therefore

$$\lim_{s \to \infty} q_s^{ij} = 0. \tag{55}$$

Allowing for (54), (55) in (29) we get

$$\lim_{s \to \infty} \left\| u^s - u^0 \right\|_{1,q} = 0. \tag{56}$$

It is easy to see that

$$\left\| \left(T_r a^s - T_r a^0\right) u^0 \right\|_2 \to 0 \text{ as } s \to \infty,$$

indeed, $|T_r a^s| \leq \frac{n}{\lambda}$ by (3) and function $\{a_{ij}(x,z,\theta)\}$, $i = 1, 2, ..., n$ are continuous by $(z,\theta) \in E_1 \times E_n$ for a.e. By $x \in D$ a.e. we'll get $g^s - g^0 \to 0$, $g^s_x - g^0_x \to 0$ a.e. $x \in D$ we'll get $T_r a^s - T_r a^0 \to 0$ a.e. $x \in D$. It remains to apply Lebesque's majorant theorem with regard to $u^0 \in L_2(D)$.

We got relation (56) for the subsequence $\{u^s\}$. Show that in fact (56) holds for all the sequence of the functions $\{u^s\}$. Let for some subsequence $\{u^{s_j}\}$ it holds

$$\lim_{s_j \to \infty} \left\| u^{s_j} - u^0 \right\|_{1,q} = \delta > 0, \qquad (57)$$

then for corresponding subsequence of $\{g^{s_j}\}$ it holds

$$\lim_{s_j \to \infty} \left\| g^{s_j} - g^0 \right\|_{1,q} = 0.$$

Repeating the previous arguments with the sequence $\{g^{s_j}\}$ we'll find its subsequence for which (56) is valid, that contradicts the supposition (57). The continuity of the operator $T : W^1_q(D) \to W^1_q(D)$ is established. Now we are to apply Schauder theorem.

Theorem 3 is proved.

References

[1]. Mamedov I.T., Mirheydarli M.M. *Neuman problem for second order nondivergent elliptic equations with discontinuous coefficients.* Proc. Of Inst. Math. and Mech. NAS of Azerbaijan, 2003, v.19(27), pp.135-152.

[2]. Fabes E., Mendes O., Marius M. *Boundary layers on Sobolev-Besov spaces and Poisson's equations for the Laplacian in Lipschitz domains.* Journal Funct. Analysis. 1998, v.159, No2, November 10, pp.322-368.

[3]. Talemti G. *Sopra una classe di eqazioni ellyptiche a coefficienti missurableili.* Ann.Math.Pure Appl., 1965, v.69, pp.285-304.

[4]. Alkhutov Yu.A., Mamedov I.T. *Some properties of the solutions of the first boundary value problem for parabolic equations with discontinuous coefficients.* Soviet Math.Dolk., 1985, v.32, No2, pp.343-347.(Russian)

[5]. Alkhutov Yu.A., Mamedov I.T. *The first boundary value problem for non-divergence second order parabolic equations with discontinuous coefficients.* Math. USSA Sbornik, 1988, v.59, No2, pp.471-495. (Russian)

[6]. Vitanra C. *$W^{2,p}$ regularity for a class of elliptic second order equations with discontinuous coefficients.* Le Mathematiche, 1992, pp.177-186.

[7]. Vitanra C. *A new contribution to the $W^{2,p}$ regularity for a class of elliptic second order equations with discontinuous coefficients.* Le mathematiche, 1993, v.48, pp.287-296

[8]. Palagachev D. *Quasilinear elliptic equations with VMO coefficients.* Trans. of Amer. Math. Soc., 1995, v. 347, No7, pp.2481-2493.

[9]. Wen G.X., Tain Mao-ying. *Initial oblique derivative problems for nonlinear parabolic equations with measurable coefficients.* Comm. In nonlinear Sci. and Num.Semin., 1998, No2, pp.109-113.

[10]. Wen G.C. *Initial-mixed boundary value problems for parabolic equations of second order with measurable coefficients in a higher dimensional domain.* Proc. of the second ISAAC Congress, 2000, v.1, pp.185-192.

[11]. Goldstein V.M., ReshtniakYu.G. *Introduction to the theory of functions with generalized derivatives and quasiconformal mappings.* M.: "Nauka", 1963. (Russian)

[12]. Ladyshenkaya O.A. *Simple proof of solvability of the main boundary value problems and the eigenvalues problem for linear elliptic equations.* Vestnik LGU, 1955, No11, pp.23-29. (Russian)

[13]. Mamedov I.T., Hassanpour S.A. *Strong solvability of the first boundary value problem for second order quasilinear elliptic equations.* Trans. NAS of Azerb., 2002, v.XXI, No4, pp.153-166.

Farman I. Mamedov, **Mirfaig M. Mirheydarli**
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.)
E-mail: farman-m@mail.ru.

Received February 20, 2004; Revised September 14, 2004.
Translated by Nazirova S.H.