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ON PALEY'S THEOREM FOR ONE SYSTEM OF EXPONENTS

Abstract

In the work the analogy of classical Paley theorem for orthonormalized system is established with regard to some nonorthogonal system of exponents with complex-valued coefficients.

Let $\{\varphi_n(t)\}_{n\geq 1}$ be some orthogonalized system and $\|\varphi_n(t)\|_\infty \leq M < +\infty$, $\forall n \geq 1$, where $\|\cdot\|$ is a norm in the space $L_\infty(a,b)$. For such systems it is well known [1, p.244] the following Paley's theorem in $L_p \equiv L_p(a,b)$.

Paley's Theorem 1. *If* $f(t) \in L_p \quad (1 < p \leq 2)$, *then*

$$\sum_{k=1}^{\infty} |a_k|^p k^{p-2} \leq A_p \int_a^b |f(t)|^p dt,$$

where A_p *doesn't depend on* $f(t)$, *and* $\{|a_k|\}$ *are the coefficients of the function* $f(t)$, *located such, that they form the sequence non-increasing by absolute value.*

2. If

$$\sum_{k=1}^{\infty} |a_k|^q k^{q-2} < +\infty \quad (q \geq 2),$$

then there exists the function $f(t) \in L_q$, for which $\{a_k\}$ is the sequence of coefficients and

$$\int_a^b |f(t)|^q dt \leq A_q \sum_{k=1}^{\infty} |a_k|^q k^{q-2},$$

where A_q doesn't depend on $\{a_k\}$, and the members of the sequence $\{a_k\}$ under the assumption and statement of the theorem don't increase by absolute value.

And now consider the following, generally speaking, non-orthonormalized system of exponents

$$\{A(t)e^{int};\ B(t)e^{ikt}\}_{n\geq 0, k\geq 1}, \tag{1}$$

with complex-valued coefficients $A(t) \equiv |A(t)|e^{i\alpha(t)}$, $B(t) \equiv |B(t)|e^{i\beta(t)}$ on the segment $[-\pi, \pi]$.

The basic properties (basicity, completeness, minimality) of this system in L_p were studied earlier in B.T.Bilalov's works. (See, for example [2,3]). It is natural that, there arises a question on the validity of the analogical theorem for the system. The present work is devoted to study of this question.

With regard to the functions $A(t)$, $B(t)$ we'll suppose the following suppositions:

1) $|A(t)|$, $|B(t)|$ are measurable, and $|A(t)|^{\pm}$, $|B(t)|^{\pm 1} \in L_\infty$.

2) $\alpha(t)$ and $\beta(t)$ are piecewise continuous functions on the segment $[-\pi; \pi]$, such that they can have the infinite number of first kind discontinuity points. Let's denote $\{\tilde{\rho}_k\} \equiv \{\alpha_k\} \cup \{\beta_k\}$, where $\{\alpha_k\}$ and $\{\beta_k\}$ are discontinuity points of the function

$\alpha(t)$ and $\beta(t)$ on the interval $(-\pi,\pi)$, respectively. The set $\{\tilde{\rho}_k\}$ can have a unique limit point $\tilde{\rho}_0 \in (-\pi,\pi)$, and the function $\theta(t) \equiv \beta(t)-\alpha(t)$ at the point $\tilde{\rho}_0$ has the finite limits from the right and the left.

3) $\sum\limits_{k=1}^{\infty}\left|\tilde{h}_i\right| < +\infty$, where $\tilde{h}_i = \theta(\tilde{\rho}_i+0)-\theta(\tilde{\rho}_i-0)$, $i=\overline{1,\infty}$ and $\tilde{h}_i \neq \dfrac{2\pi}{p}+2\pi k$ for any integer k.

Denote by r the number, after which it is fulfilled

$$-\frac{2\pi}{q} < \tilde{h}_i < \frac{2\pi}{p},\ k=\overline{r,\infty}, \tag{2}$$

where $q : \dfrac{1}{p}+\dfrac{1}{q}=1$ is an adjoint number.

Let's renumber the elements of the set $\{\tilde{\rho}_i\}_0^r$ according to increase and denote by $\{\rho_i\}_0^r$. Renumber the corresponding to them jumps $\{\tilde{h}_i\}_0^r$ and denote by $\{h_i\}_0^r$ where $\tilde{h}_0 = \theta(\tilde{\rho}_0+0)-\theta(\tilde{\rho}_0-0)$. Define the integers $n_i,\ i=\overline{1,r}$ from the following inequalities

$$-\frac{1}{q} < \frac{h_i}{2\pi}+n_{i-1}-n_i < \frac{1}{p},\ i=\overline{1,r}, \tag{3}$$

$$n_0 = 0.$$

Let

$$\omega = \frac{1}{2\pi}\left[\beta(-\pi)-\beta(\pi)+\alpha(\pi)-\alpha(-\pi)\right]+n_r. \tag{4}$$

It holds the following

Theorem. *Let the quantity ω defined from correlations (2)-(4) satisfy the inequality*

$$-\frac{1}{q} < \omega < \frac{1}{p}.$$

Then: 1) If $f(t) \in L_p\quad(1<p\leq 2)$, then

$$\sum_{k=1}^{\infty}\left|a_{k-1}^{+}\right|^p k^{p-2} + \sum_{k=1}^{\infty}\left|a_k^{-}\right|^p k^{p-2} \leq M_p\int_{-\pi}^{\pi}|f(t)|^p\,dt,$$

where $\{a_k^{+};a_{k+1}^{-}\}_{k\geq 0}$ are biorthogonal coefficients by system (1) of the function $f(t)$, and the constant M_p doesn't depend on f.

2) If the subsequence $\{a_k^{+};a_{k+1}^{-}\}_{k\geq 0}$ is such that

$$\sum_{k=1}^{\infty}\left(\left|a_{k-1}^{+}\right|^p+\left|a_k^{-}\right|^p\right)k^{p-2} < +\infty$$

for $p\geq 2$, then there exists the function $f(t)\in L_p$, for which $\{a_k^{\pm}\}$ is a sequence of biorthogonal coefficients by system (1) and

$$\int_{-\pi}^{\pi}|f(t)|^p\,dt \leq M_p'\sum_{k=1}^{\infty}\left(\left|a_{k-1}^{+}\right|^p+\left|a_k^{-}\right|^p\right)k^{p-2},$$

where constant M_p' doesn't depend on $\{a_k^{\pm}\}$.

Proof. Let's take arbitrary function $f(t) \in L_p(-\pi,\pi)$, where $p \in (1,2]$ is some number. Consider the following Riemann's problem of the theory of analytical functions in the Hardy classes $H_p^{\pm}$:

$$\begin{cases} F^+\left(e^{it}\right) + G(t)F^-\left(e^{it}\right) = g(t), \\ F^-(\infty) = 0, \ -\pi < t < \pi, \end{cases} \tag{5}$$

where $g(t) \equiv \dfrac{f(t)}{A(t)}$. Under the solution of the adjoint problem (5) we understand the following: the pair of analytical functions $F^+(z)$ and $F^-(z)$, belonging to the classes H_p^+ and H_p^- respectively, whose non-tangent boundary values satisfy condition (5) almost everywhere on a unit circle, is searched. The theory of such problems is well developed (see, for example [4]). From conditions of the theorem and from the results of the paper [2] it follows, that problem (5) in Hardy's classes $H_p^{\pm}$ has a unique solution, which vanishes at infinity. Let's denote by $\{h_n^+(t); h_{n+1}^-(t)\}_{n\geq 0}$ a system biorthogonal to (1) i.e.

$$\int_{-\pi}^{\pi} A(t)\, e^{int}\overline{h_k^+(t)}dt = \delta_{nk}, \qquad \int_{-\pi}^{\pi} A(t)\, e^{int}\overline{h_k^-(t)}dt = 0,$$

$$\int_{-\pi}^{\pi} B(t)\, e^{-int}\overline{h_k^+(t)}dt = 0, \qquad \int_{-\pi}^{\pi} B(t)\, e^{-int}\overline{h_k^-(t)}dt = \delta_{nk},$$

where δ_{nk} is a Kroneker's symbol $\left(\overline{\cdot}\right)$ is a complex conjunction. Let $\{a_n^+; a_{n+1}^-\}_{n\geq 0}$ be biorthogonal coefficients of the function $f(t)$ by system (1):

$$a_n^{\pm} = \int_{-\pi}^{\pi} f(t)\, \overline{h_k^+(t)}dt.$$

As is shown in the paper [2], the biorthogonal coefficients $\{a_n^+\}_{n\geq 0}$, $\{a_n^-\}_{n\geq 1}$ of the functions f by system (1) are Fourier coefficients of the functions $F^+\left(e^{it}\right)$ and $F^-\left(e^{it}\right)$, respectively, by classical system of exponents $\{e^{int}\}_{-\infty}^{+\infty}$:

$$a_n^+ = \int_{-\pi}^{\pi} f(t)\, \overline{h_n^+(t)}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^+\left(e^{it}\right) e^{-int}dt, \quad n \geq 0$$

and

$$a_n^- = \int_{-\pi}^{\pi} f(t)\, \overline{h_n^-(t)}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^-\left(e^{it}\right) e^{int}dt, \quad n \geq 1.$$

From the belonging of the functions $F^+\left(e^{it}\right)$, $F^-\left(e^{it}\right)$ to the classes H_p^+ and H_p^-, respectively, it follows:

$$a_n^+ = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^+\left(e^{it}\right) e^{-int}dt = 0, \quad \forall n \leq -1,$$

$$a_n^- = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^-\left(e^{it}\right)e^{int}dt = 0, \quad \forall n \le 0.$$

Then from the named theorem P we directly get:

$$\sum_{n=1}^{\infty}\left|a_{n-1}^+\right|^p n^{p-2} \le A_p\int_{-\pi}^{\pi}\left|F^+\left(e^{it}\right)\right|^p dt,$$

$$\sum_{n=1}^{\infty}\left|a_n^-\right|^p n^{p-2} \le A_p\int_{-\pi}^{\pi}\left|F^-\left(e^{it}\right)\right|^p dt.$$

By summing we have:

$$\left(\sum_{n=1}^{\infty}\left|a_{n-1}^+\right|^p n^{p-2}\right)^{1/p} + \left(\sum_{n=1}^{\infty}\left|a_n^-\right|^p n^{p-2}\right)^{1/p} \le A_p\left[\left\|F^+\left(e^{it}\right)\right\|_p + \left\|F^-\left(e^{it}\right)\right\|_p\right],$$

where $\|\cdot\|_p$ is the norm in $L_p(-\pi,\pi)$. As a result, allowing for Minkovskii inequality we get

$$\left(\sum_{n=1}^{\infty}\left[\left|a_{n-1}^+\right|^p + \left|a_n^-\right|^p\right]n^{p-2}\right)^{1/p} \le A_p\left(\left\|F^+\left(e^{it}\right)\right\|_p + \left\|F^-\left(e^{it}\right)\right\|_p\right). \qquad (6)$$

As a result from (6) we have

$$\left(\sum_{n=1}^{\infty}\left[\left|a_{n-1}^+\right|^p + \left|a_n^-\right|^p\right]n^{p-2}\right)^{1/p} \le C_p\left(\left\|A(t)\,F^+\left(e^{it}\right)\right\|_p + \left\|B(t)\,F^-(t)\right\|_p\right). \quad (7)$$

Here we used that the functions $A(t)$ and $B(t)$ satisfy the condition 1).

From the basicity of system (1) in L_p and basicity criterian [5] the following obvious inequalities follow:

$$\left\|A(t)\,F^+\left(e^{it}\right)\right\|_p \le B_p\,\|f\|_p\,,$$

$$\left\|B(t)\,F^-\left(e^{it}\right)\right\|_p \le B_p\,\|f\|_p\,.$$

Considering these inequalities in (7) we get the desired.

Now, let's pass to the proof of the second part of the theorem. So, let the sequence $\{a_n^+; a_{n+1}^-\}_{n\ge 0}$ be such that

$$\sum_{n=1}^{\infty}\{\left|a_{n-1}^+\right|^p + \left|a_n^-\right|^p\}n^{p-2} < +\infty.$$

Then by theorem P

$$F^+\left(e^{it}\right) = \sum_{n=0}^{\infty}a_n^+e^{int}$$

and

$$F^{-}\left(e^{it}\right)=\sum_{n=1}^{\infty}a_n^{-}e^{-int}$$

represent some functions from L_p, where

$$\int_{-\pi}^{\pi}\left|F^{+}\left(e^{it}\right)\right|^p dt\leq A_p\sum_{n=1}^{\infty}\left|a_{n-1}^{+}\right|^p n^{p-1}$$

and

$$\int_{-\pi}^{\pi}\left|F^{-}\left(e^{it}\right)\right|^p dt\leq A_p\sum_{n=1}^{\infty}\left|a_n^{-}\right|^p n^{p-4}.$$

Denoting

$$f=A\left(t\right)F^{+}\left(e^{it}\right)+B\left(t\right)F^{-}\left(e^{it}\right),$$

we get, that sequences of the numbers $\{a_{n-1}^{+};a_n^{-}\}_{n\geq 1}$ are biorthogonal coefficients of the function $f\left(t\right)$ by system (8). From the conditions of the theorem, applainl Minkovsii's inequality and taking into account Reisz property [6, p.113]

$$\left\|\sum_{k=0}^{m}a_k e^{ikt}\right\|_p+\left\|\sum_{k=-m}^{-1}a_k e^{ikt}\right\|_p\leq C\left\|\sum_{k=-m}^{m}a_k e^{ikt}\right\|_p$$

we get

$$\|f\|_p\leq M_1\left(\left\|F^{+}\left(e^{it}\right)\right\|_p+\left\|F^{-}\left(e^{it}\right)\right\|_p\right)\leq M_2\left\|F^{+}\left(e^{it}\right)+F^{-}\left(e^{it}\right)\right\|_p. \qquad (8)$$

It is obvious that $\{a_{n-1}^{+};a_n^{-}\}_{n\geq 1}$ are the Fourier coefficients of the functionm $F^{+}\left(e^{it}\right)+F^{-}\left(e^{it}\right)$ by a classical system of exponents $\{e^{int}\}_{-\infty}^{+\infty}$. Then from (8) using the theorem P we get the validity of the affirmation of the second part of the theorem. The theorem is proved. The following corollary follows from this theorem.

Corollary. Let a complex parameter α satisfy the inequality

$$-\frac{1}{2q}<\operatorname{Re}\alpha<\frac{1}{2p}.$$

Then for the system of exponents

$$1\cup\left\{\exp\left[i\left(n+\alpha\ sign\ n\right)\right]\right\}_{n=\pm 1,\ldots}; \qquad (9)$$

it is valid:

1) If $f\left(t\right)\in L_p\quad\left(1<p\leq 2\right)$, then

$$\sum_{-\infty}^{+\infty}\left|a_k\right|^p\cdot\left|k\right|^{p-2}\leq C_p\int_{-\pi}^{\pi}\left|f\left(t\right)\right|^p dt,$$

where $\{a_k\}_{-\infty}^{+\infty}$ are biorthogonal coefficients of the function $f\left(t\right)$ by system (9), the constant C_p depends only on p,

2) If sequence $\{a_k\}_{-\infty}^{+\infty}$ is such, that

$$\sum_{-\infty}^{+\infty} |a_k|^p \cdot |k|^{p-2} < +\infty$$

for $p \geq 2$, then there exists the function $f \in L_p$, for which $\{a_k\}_{-\infty}^{+\infty}$ are biorthogonal coefficients and

$$\int_{-\pi}^{\pi} |f(t)|^p \, dt \leq C'_p \sum_{-\infty}^{+\infty} |a_k|^p \cdot |k|^{p-2},$$

where constant C'_p depends only on p and under $|0|^{p-2}$ we have a unit.

Let's mark, that basic properties of system (9) were studied by many authors beginning from the paper [7].

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