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ON SOLVABILITY OF ONE BOUNDARY-VALUE PROBLEM FOR A SECOND ORDER OPERATOR-DIFFERENTIAL EQUATION ON A BAND

Abstract

The sufficient conditions, providing the regular solvability of some boundary-value problem for second order operator-differential equation on the band are obtained. These conditions are expressed in terms of coefficients of the given operator-differential equation.

Let H be a separable Hilbert space, A be a normal reversible operator in H. Then A has a polar expansion $A = U\,|A|$, where U is a unitary operator in H, and $|A|$ is a positive definite selfadjoint operator in H.

Let us denote by H_α the scale of Hilbert spaces generated by the operator A i.e., $H_\alpha = D\left(|A|^\alpha\right),\ (\varphi,\psi)_\alpha = \left(|A|^\alpha \varphi, |A|^\alpha \psi\right),\ \varphi,\psi \in D\left(|A|^\alpha\right)$.

Let $x \in R^1 = (-\infty,\infty),\ t \in (0,1)$. Consider in the band $Q = (0,1) \times R^1$ the boundary-value problem

$$-\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + A^2 u + A_{1,0}\frac{\partial u}{\partial t} + A_{0,1}\frac{\partial u}{\partial x} + A_{0,0}u = f,\ (t,x) \in Q, \tag{1}$$

$$u\,(0,x) = u\,(1,x) = 0, \tag{2}$$

where $f\,(t,x) \in L_2\,(Q,H)\,,\ u\,(t,x) \in W_{2,2}^2\,(Q,H)$, and the operator coefficients satisfy the following conditions:

1) A is a normal reversible operator, whose spectrum is contained in the corner sector

$$S_\varepsilon = \{\lambda : |\arg\lambda| \le \varepsilon\},\ 0 \le \varepsilon < \frac{\pi}{2}.$$

2) The operators $B_{1,0} = A_{1,0}A^{-1},\ B_{0,1} = A_{0,1}A^{-1}, B_{0,0} = A_{0,0}A^{-2}$ are bounded in H.

Let's note, that

$$L_2\,(Q,H) = \left\{ f : \|f\|_{L_2(Q;H)} = \int_0^1\int_{-\infty}^{\infty} \|f\,(t,x)\|^2\, dtdx < \infty \right\},$$

and $W_{2,2}^2(Q,H)$ is a Hilbert space of vector-functions, obtained by completion of infinitely-differentiable functions with values from H_2, which have the compact supports in Q with the norm

$$\|u\|_{W_{2,2}^2(Q;H)} =$$

$$= \left(\left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L_2(Q;H)} + \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|^2_{L_2(Q;H)} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|^2_{L_2(Q;H)} + \left\| A^2 u \right\|^2_{L_2(Q;H)} \right)^{1/2}.$$

Denote by

$$\mathring{W}_2^2(Q;H) = \left\{ u \left| u \in W_2^2(Q;H),\ u(0,x) = u(1,x) = 0 \right. \right\}.$$

Then we'll define spaces [1]

$$W_2^2((0,1);H) = \{\nu : \nu'' \in L_2((0,1);H),\ A^2\nu \in L_2((0,1);H)\},$$

and

$$\mathring{W}_2^2((0,1);H) = \{\nu : \nu \in W_2^2((0,1);H),\ \nu(0) = \nu(1) = 0\}$$

with the norm

$$\|\nu\|_{W_2^2((0,1);H)} = \left(\|\nu''\|^2_{L_2((0,1);H)} + \|A^2\nu\|^2_{L_2((0,1);H)} \right)^{1/2}.$$

Definition 1. *If at any $f(t,x) \in L_2(Q;H)$ there exists the vector-function $u(t,x) \in W_{2,2}^2(Q;H)$ which satisfies equation (1) almost everywhere in Q, boundary condition (2) in the sense*

$$\lim_{t\to 0} \|u(t,x)\|_{3/2} = 0,\ \lim_{t\to 1} \|u(t,x)\|_{3/2} = 0$$

and inequality

$$\|u\|_{W_{2,2}^2(Q;H)} \le const\, \|f\|_{L_2(Q;H)},$$

then problem (1), (2) we'll call a regular solvable.

In the given paper we'll find the conditions for regular solvability of problem (1), (2).

Let's note, that in one-dimensional case the analogical problem is considered in the paper [2].

Let

$$P_0 u = -\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + A^2 u,\ u \in W_{2,2}^2(Q;H), \tag{3}$$

$$P_1 u = A_{1,0} \frac{\partial u}{\partial t} + A_{0,1} \frac{\partial u}{\partial x} + A_{0,0} u,\ u \in W_{2,2}^2(Q;H). \tag{4}$$

Lemma 1. *Let condition (1) be fulfilled. Then for $\xi \in R$ the operator*

$$L_0(\xi) = -\frac{\partial^2 \nu}{\partial t^2} + \left(\xi^2 E + A^2\right)\nu, \nu \in \mathring{W}_2^2((0,1);H) \tag{5}$$

maps the space $\overset{\circ}{W}{}_2^2((0,1);H)$ *onto* $L_2((0,1);H)$.

Proof. For $\xi\in R$ the spectrum of the normal operator $(\xi^2E+A^2)^{1/2}$ is contained in the corner sector $S_\varepsilon=\{\lambda:|\arg\lambda|\le\varepsilon\}$, $0\le\varepsilon<\frac{\pi}{2}$. The general solution of the equation $L_0(\xi)\nu=0$ from the space $W_2^2((0,1);H)$ has the form

$$\nu_0(t)=e^{-(\xi^2E+A^2)^{1/2}}\varphi_0+e^{(\xi^2E+A^2)^{1/2}(t-1)}\varphi_1,$$

where $\varphi_0,\varphi_1\in H_{3/2}$. From the condition $\nu_0(t)\in\overset{\circ}{W}{}_2^2((0,1);H)$ if follows, that

$$\varphi_0+e^{-(\xi^2E+A^2)^{1/2}}\varphi_1=0,$$

$$e^{-(\xi^2E+A^2)^{1/2}}\varphi_0+\varphi_1=0.$$

Hence, we obtain that $\varphi_0=\varphi_1=0$, i.e. $\nu_0(t)=0$. But at any $g(t)\in L_2((0,1);H)$ the equation $L_0(\xi)\nu=g$ has a solution from the space $\overset{\circ}{W}{}_2^2((0,1);H)$.

Really,

$$\nu_1(t,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\xi^2E+\eta^2E+A^2\right)^{-1}\int_0^1 g(s)\,e^{i\eta(t-s)}dsdt,\ t\in R$$

satisfies the equation $L_0(\xi)\nu=g$ almost everywhere. From the Plansharel theorem, it is easy to obtain, that $\nu_1(t,\xi)\in L_2(R;H)$.

Really,

$$\|\nu_1\|^2_{W_2^2(R;H)}+\left\|A^2\nu_1\right\|^2_{W_2^2(R;H)}=\left\|\eta^2\left(\eta^2E+\xi^2E+A^2\right)^{-1}\hat{g}(\tau)\right\|^2_{L_2(R;H)}+$$

$$+\left\|A^2\left(\eta^2E+A^2\right)^{-1}\hat{g}(\tau)\right\|^2_{L_2(R;H)}\le\sup_{\eta\in R}\left\|\eta^2\left(\eta^2E+\xi^2E+A^2\right)^{-1}\right\|^2\|g\|^2_{L_2}+$$

$$+\sup_{\eta\in R}\left\|A^2\left(\eta^2E+\xi^2E+A^2\right)^{-1}\right\|^2\|g\|^2_{L_2}.$$

From the spectral expansion of the operator A it follows, that

$$\sup_{\eta\in R}\left\|A^2\left(\eta^2E+\xi^2E+A^2\right)^{-1}\right\|\le c_0(\varepsilon),\ \sup_{\eta\in R}\left\|\eta^2\left(\eta^2E+\xi^2E+A^2\right)^{-1}\right\|c_0(\varepsilon),$$

where

$$c_0(\varepsilon)=\begin{cases}1, 0\le\varepsilon\le\frac{\pi}{4},\\[2ex] \frac{1}{\sqrt{2}\cos\varepsilon},\frac{\pi}{4}\le\varepsilon<\frac{\pi}{2}.\end{cases}$$

So, $\nu_1(t,\xi)\in L_2(R;H)$. Denote contraction of the vector-function $\nu_1(t,\xi)$ on $[0,1]$ by $\overline{\nu}_1(t,\xi)$. Then the general solution of the equation $L_0(\xi)\nu=g$ from the space $\overset{\circ}{W}{}_2^2((0,1);H)$ has the form:

$$\nu(t,\xi)=\overline{\nu}_1(t,\xi)+e^{-(\xi^2E+A^2)^{1/2}t}\varphi_0+e^{(\xi^2E+A^2)^{1/2}(t-1)}\varphi_1,$$

where the vectors $\varphi_0, \varphi_1 \in H_{3/2}$ are to be defined. From the condition $\nu(0) = \nu(1) = 0$ it follows, that

$$\begin{cases} \varphi_0 + e^{-\left(\xi^2 E + A^2\right)^{1/2} t} \varphi_1 = -\overline{\nu}_1(0,\xi), \\ e^{-\left(\xi^2 E + A^2\right)^{1/2} t} + \varphi_1 = -\overline{\nu}_2(0,\xi). \end{cases}.$$

Solving this system and using the fact that $\overline{\nu}_1(0,\xi), \ \overline{\nu}_2(0,\xi) \in H_{3/2}$ we'll uniquely define $\varphi_0, \varphi_1 \in H_{3/2}$, hence and $\nu(t,\xi)$. The statement of the lemma follows from the Banach theorem on the inverse operator.

Lemma 2. *At any $\xi \in R$ and $\nu \in \overset{\circ}{W}{}_2^2((0,1);H)$ there hold the inequalities*

$$1) \ \|L_0(\xi)\nu\|^2_{L_2((0,1);H)} \geq \left\|\left(\xi^2 E + A^2\right)\nu\right\|^2_{L_2((0,1);H)} +$$

$$+2\cos 2\varepsilon \left\|\left(\xi^2 E + A^2\right)^{1/2} \nu'\right\|^2_{L_2((0,1);H)}, \tag{6}$$

$$2) \ \left\|A^2\nu\right\|_{L_2((0,1);H)} \leq c_0^2(\varepsilon)\,\|L_0(\xi)\|^2_{L_2((0,1);H)} \tag{7}$$

$$3) \ \left\|A\frac{\partial \nu}{\partial t}\right\|_{L_2((0,1);H)} \leq c_0^{1/2}(\varepsilon)\,c_1(\varepsilon)\,c_1(\varepsilon)\,\|L_0(\xi)\nu\|_{L_2((0,1);H)}, \tag{8}$$

$$4) \ \|\xi A\nu\|_{L_2((0,1);H)} \leq c_0(\varepsilon)\,c_1(\varepsilon)\,\|L_0(\xi)\nu\|_{L_2((0,1);H)} \tag{9}$$

where

$$c_0(\varepsilon) = \begin{cases} 1, 0 \leq \varepsilon \leq \frac{\pi}{4} \\ \frac{1}{\sqrt{2}\cos\varepsilon}, \ \frac{\pi}{4} \leq \varepsilon < \frac{\pi}{2}. \end{cases} \tag{10}$$

$$c_1(\varepsilon) = \frac{1}{2\cos\varepsilon}, \ \varepsilon \in \left[0, \frac{\pi}{2}\right). \tag{11}$$

Proof. For $\xi \in R, \ \nu \in \overset{\circ}{W}{}_2^2((0,1);H)$

$$\|L_0(\xi)\nu\|^2_{L_2((0,1);H)} = \left\|\left(\xi^2 E + A^2\right)\nu\right\|^2_{L_2((0,1);H)} + \left\|\nu''\right\|^2_{L_2((0,1);H)} -$$

$$-2\,\mathrm{Re}\left(\left(\xi^2 E + A^2\right)\nu, \nu''\right)_{L_2((0,1);H)}.$$

But taking into account, that $\nu(0) = \nu(1) = 0$ and the spectrum of the operator $\left(\xi^2 E + A^2\right)^{1/2}$ is contained in the corner spectrum S_ε, after integration by parts we'll obtain:

$$-2\,\mathrm{Re}\left(\left(\xi^2 E + A^2\right)\nu, \nu''\right)_{L_2((0,1);H)} =$$

$$= 2\,\mathrm{Re}\left(\left(\xi^2 E + A^2\right)^{1/2}\nu', \left(\xi^2 E + A^{*2}\right)^{1/2}\nu'\right)_{L_2((0,1);H)} \geq$$

$$\geq 2\cos 2\varepsilon \left\| \left(\xi^2 E + A^2\right)^{1/2} \right\|^2_{L_2((0,1);H)}.$$

Therefore, inequality (6) is true.

Let's prove the rest inequalities. At any $\xi \in R$ the operator $\left(\xi^2 E + A^2\right)^{1/2}$ we'll represent in the form $U(\xi)\left|\left(\xi^2 E + A^2\right)^{1/2}\right|$, where $U(\xi)$ is a unitary operator, and $\left|\left(\xi^2 E + A^2\right)^{1/2}\right|$ is a positive operator in H. Then for $u \in \overset{\circ}{W}{}_2^2((0,1);H)$ it holds the inequality

$$\left\| \left(\xi^2 E + A^2\right)^{1/2} \nu' \right\|^2_{L_2((0,1);H)} = \int_0^1 \left(\left| \left(\xi^2 E + A^2\right)^{1/2} \right| \nu', \left| \left(\xi^2 E + A^2\right)^{1/2} \right| \nu' \right) dt =$$

$$= -\int_0^1 \left(\left| \xi^2 E + A^2 \right| \nu, \nu'' \right) dt \leq \left\| \left| \xi^2 E + A^2 \right| \nu \right\|_{L_2((0,1);H)} \left\| \nu'' \right\|_{L_2((0,1);H)} \leq$$

$$\leq \frac{1}{2} \left(\left\| \left(\xi^2 E + A^2\right) \nu \right\|^2_{L_2((0,1);H)} + \left\| \nu'' \right\|^2_{L_2((0,1);H)} \right).$$

Taking into account this inequality in (6) we have

$$\left\| \left(\xi^2 E + A^2\right)^{1/2} \nu' \right\|^2_{L_2((0,1);H)} \leq \frac{1}{2(1+\cos 2\varepsilon)} \left\| L_0(\xi)\nu \right\|^2_{L_2((0,1);H)} =$$

$$= \frac{1}{4\cos^2\varepsilon} \left\| L_0(\xi)\nu \right\|^2_{L_2((0,1);H)}.$$

Hence, it follows that

$$\left\| \left(\xi^2 E + A^2\right)^{1/2} \nu' \right\|^2_{L_2((0,1);H)} \leq \frac{1}{2(1+\cos 2\varepsilon)} \left\| L_0(\xi)\nu \right\|^2_{L_2((0,1);H)} =$$

$$= \frac{1}{4\cos^2\varepsilon} \left\| L_0(\xi)\nu \right\|^2_{L_2((0,1);H)}$$

i.e.

$$\left\| \left(\xi^2 E + A^2\right)^{1/2} \nu' \right\|_{L_2((0,1);H)} \leq \frac{1}{2\cos\varepsilon} \left\| L_0(\xi)\nu \right\|_{L_2((0,1);H)}. \tag{12}$$

On the other hand

$$\left\| A\nu' \right\|_{L_2((0,1);H)} \leq \sup_{\xi\in R} \left\| A\left(\xi^2 E + A^2\right)^{-1/2} \right\| \cdot \left\| \left(\xi^2 E + A^2\right)^{1/2} \nu' \right\|_{L_2((0,1);H)}.$$

Since

$$\left\| A\left(\xi^2 + A^2\right)^{-1/2} \right\| \leq \sup_{\mu>0, |\varphi|\leq\varepsilon} \left| \mu \left(\mu^2 e^{2i\varphi} + \xi^2\right)^{-1/2} \right| =$$

$$= \sup_{\mu>0} \left| \mu \left(\mu^4 + \xi^4 + 2\xi^2\mu^2\cos 2\varepsilon\right)^{-1/4} \right| \leq c_0^{1/2}(\varepsilon),$$

where $c_0(\varepsilon)$ is defined from equality (10). Thus,

$$\left\| A\nu' \right\|_{L_2((0,1);H)} \leq c_0^{1/2}(\varepsilon)\, c_1(\varepsilon) \left\| L_0(\xi)\nu \right\|_{L_2((0,1);H)}.$$

Inequality (8) is proved. Then from inequality (6) we obtain, that for $0 \le \varepsilon \le \dfrac{\pi}{4}$

$$\left\| \left(A^2 + \xi^2 E\right) \nu \right\|_{L_2((0,1);H)} \le \left\| L_0 (\xi) \nu \right\|_{L_2((0,1);H)}, \qquad (13)$$

and for $\dfrac{\pi}{4} \le \varepsilon < \dfrac{\pi}{2}$ from inequality (6) allowing for inequality (12) we obtain

$$\left\| \left(A^2 + \xi^2 E\right) \nu \right\|_{L_2((0,1);H)} \le \frac{1}{2\cos^2 \varepsilon} \left\| L_0 (\xi) \nu \right\|^2_{L_2((0,1);H)}. \qquad (14)$$

From (13) and (14) it follows, that

$$\left\| \left(A^2 + \xi^2 E\right) \nu \right\|_{L_2((0,1);H)} \le c_0 (\varepsilon) \left\| L_0 (\xi) \nu \right\|_{L_2((0,1);H)}.$$

Then

$$\left\| A^2 \nu \right\|_{L_2((0,1);H)} \le \sup_{\xi \in R} \left\| A \left(A^2 + \xi^2 E\right)^{-1} \right\| \cdot \left\| \left(A^2 + \xi^2 E\right) \nu \right\|_{L_2((0,1);H)} \le$$
$$\le c_0^2 (\varepsilon) \left\| L_0 (\xi) \nu \right\|_{L_2((0,1);H)}.$$

Inequality (7) is also proved. We'll prove inequality (9).

Since

$$\left\| \xi A \nu \right\|_{L_2((0,1);H)} \le \sup_{\xi \in R} \left\| \xi A \left(\xi^2 E + A^2\right)^{-1} \right\| \left\| \left(\xi^2 E + A^2\right) \nu \right\|_{L_2((0,1);H)} \le$$
$$\le \sup_{\xi \in R} \sup_{\mu > 0, |\varphi| \le \varepsilon} \left| \xi \mu \left(\mu^2 e^{2i\varphi} + \xi^2\right)^{-1} \right| \cdot \left\| \left(\xi^2 E + A^2\right) \nu \right\|_{L_2((0,1);H)} \le$$
$$\le c_1 (\varepsilon) \left\| \left(\xi^2 E + A^2 \nu\right) \right\|_{L_2((0,1);H)} \le c_1 (\varepsilon) c_0 (\varepsilon) \left\| L_0 (\xi) \nu \right\|_{L_2((0,1);H)}.$$

Lemma 2 is proved.

Theorem 1. *The operator* $P_0 : \overset{\circ}{W}{}_2^2 (Q; H) \to L_2 (Q; H)$ *is an isomorphism.*

Proof. After Fourier transformation on the variable x from the equation $P_0 u = f$ we obtain $L_0 (\xi) \hat{u} (t, \xi) = \hat{f} (t, \xi)$. At $\hat{f} (t, \xi) \equiv 0$, the equation $L_0 (\xi) \hat{u} (t, \xi) = 0$ has only zero solution, i.e. $\hat{u} (t, \xi) = 0$. Hence, the equation $P_0 u = 0$ has only zero regular solution. On the other hand the equation $P_0 u = f$ has a regular solution at any $f \in L_2 (Q; H)$, since the equation $L_0 (\xi) \hat{u} (t, \xi) = \hat{f} (t, x)$ is solvable at any $\hat{f} (t, \xi) \in L_2 ((0, 1); H), \xi \in R$. Then we can find $u (t, x)$.

The theorem is proved.

Theorem 2. *Let the conditions 1) and 2) be fulfilled, at that*

$$\alpha (\varepsilon) = c_0^{1/2} (\varepsilon) c_1 (\varepsilon) \left\| B_{1,0} \right\| + c_0 (\varepsilon) c_1 (\varepsilon) \left\| B_{0,1} \right\| + c_0^2 (\varepsilon) \left\| B_{0,0} \right\| < 1.$$

Then problem (1), (2) is regular solvable.

Proof. We'll write equations (1) in the form

$$P_0 u + P_1 u = f, \ u \in \overset{\circ}{W}{}_2^2 (Q; H), \ f \in L_2 (Q; H).$$

By theorem 1 the operator P_0 has the bounded inverse. Then after substitution $u = P_0^{-1}\omega$ we obtain the equation $\left(E + P_1 P_0^{-1}\right)\omega = f$ in the space $L_2(Q;H)$.

Since

$$\left\|P_1 P_0^{-1}\omega\right\|_{L_2(Q;H)} = \|P_1 u\|_{L_2(Q;H)} \leq$$

$$\leq \|B_{1,0}\| \left\|A\frac{\partial u}{\partial t}\right\|_{L_2(Q;H)} + \|B_{0,1}\| \left\|A\frac{\partial u}{\partial x}\right\|_{L_2(Q;H)} + \|B_{0,0}\| \left\|A^2 u\right\|_{L_2(Q;H)}. \qquad (15)$$

Using Plansharel theorem from Lemma 2 we get

$$\left\|A\frac{\partial u}{\partial t}\right\|_{L_2(Q;H)} = \left\|A\frac{\partial \hat{u}(t,\xi)}{\partial t}\right\|_{L_2(Q;H)} \leq c_0^{1/2}(\varepsilon)\cdot c_1(\varepsilon)\left\|L_2(\xi)\,\hat{u}(t,\xi)\right\|_{L_{(Q;H)}} =$$

$$= c_0^{1/2}(\varepsilon)\cdot c_1(\varepsilon)\|P_0 u\|_{L_2(Q;H)} = c_0^{1/2}(\varepsilon)\cdot c_1(\varepsilon)\|\omega\|_{L_2(Q;H)}. \qquad (16)$$

Similarly, there hold the inequalities

$$\left\|A\frac{\partial u}{\partial x}\right\|_{L_2(Q;H)} \leq c_0(\varepsilon)\cdot c_1(\varepsilon)\|\omega\|_{L_2(Q;H)} \qquad (17)$$

and

$$\left\|A^2 u\right\|_{L_2(Q;H)} \leq c_0^2(\varepsilon)\|\omega\|_{L_2(Q;H)}. \qquad (18)$$

From inequalities (15), (16), (17) and (18) it follows, that

$$\left\|P_1 P_0^{-1}\right\|_{L_2(Q;H)} \leq \alpha(\varepsilon) < 1.$$

Then

$$u = P_0^{-1}\left(E + P_1 P_0^{-1}\right)^{-1} f.$$

The theorem is proved.

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