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ON FOURFOLD COMPLETENESS OF ROOT VECTORS OF ONE CLASS OF FOURTH ORDER OPERATOR BUNDLES DEPENDING ON PARAMETERS

Abstract

In this paper the principles of four-fold completeness of eigen and adjoined vectors of fourth order operator bundles depending on parameters are established. Under some values of parameters the characteristic polynomial of the leading part of operator bundle has a multiple root. At fulfilling some algebraic conditions imposed on smallness of relative norms of coefficients the theorems on fourfold completeness of eigen and adjoined vectors are obtained, and their dependences on parameters is established.

Consider in the separable Hilbert space H the following fourth order polynomial operator bundle depending on two parameters – on spectral parameter $\lambda \in C$ and real parameter $\varepsilon \in [0;1]$

$$L(\lambda;\varepsilon) = \lambda^4 E - 2\varepsilon\lambda^2 G^2 + G^4 + \sum_{j=0}^{3} \lambda^j F_{4-j}, \qquad (1)$$

where the following conditions are supposed for the coefficients of this bundle:

1) G is a positive-definite selfadjoint operator;

2) $G^{-1} \in \sigma_p \quad (0 < p < \infty)$;

3) The operators $F_j G^{-j}$ $(j = 1, 2, 3, 4)$ are bounded in H;

4) The operator $G^4 + F_4$ is reversible in H.

At fulfilling conditions 1)-4) the operator bundle $L(\lambda;\varepsilon)$ (see [1]) has a discrete spectrum at each $\varepsilon \in [0;1]$.

It is obvious that at $\varepsilon \in [0;1)$ the leading part of operator bundle (1)

$$L_0(\lambda;\varepsilon;G) = \lambda^4 E - 2\varepsilon\lambda^2 G^2 + G^4$$

has a simple spectrum, in the sense that the characteristic polynomial

$$L_0(\lambda;\varepsilon;\mu) = \lambda^4 - 2\varepsilon\lambda^2\mu^2 + \mu^4, \quad \mu \in \sigma(G)$$

at $\varepsilon \in [0;1)$ has simple roots, and at $\varepsilon = 1$ the leading part $L_0(\lambda;\varepsilon;G)$ has a multiple spectrum.

In this paper we find the conditions on coefficients of operator bundle (1) which provide fourfold completeness of all eigen and adjoined vectors of operator bundle $L(\lambda;\varepsilon)$ at $\varepsilon \in [0;1]$.

Note that at $\varepsilon = 0$ operator bundle (1) has been considered in the paper [1] when $F_j G^{-j}$ $(j = 1, 2, 3, 4)$ are completely continuous operators. It has been considered and in [2] when G is a normal reversible operator, F_4G^{-4} is a bounded operator with sufficient small norm, and F_jG^{-j} $(j=1,2,3)$ are completely continuous operators.

Definition 1. *Let at some fixed* $\varepsilon\in[0;1]$ *there exists* $\varphi_0\neq 0$ *which satisfies the equation*

$$L_0(\lambda_0;\varepsilon)\varphi_0=0.$$

Then λ_0 *is called an eigen number and* φ_0 *is called an eigen vector of the bundle* $L_0(\lambda;\varepsilon)$. *The system* $\varphi_1,...,\varphi_m$ *is called adjoined to the vector* φ_0 *if these vectors satisfy the equations*

$$\sum_{k=0}^{4}\frac{L^{(k)}(\lambda_0;\varepsilon)}{k!}\varphi_{q-k}=0,\quad q=0,1,...,m. \tag{2}$$

Let φ_0 be an eigen vector, and $\varphi_1,...,\varphi_m$ be its adjoined vectors corresponding to the eigen value λ_0. Then the vector functions

$$u_p(\lambda_0,t)=e^{\lambda_0 t}\left(\frac{t^p}{p!}\varphi_p+\frac{t^{p-1}}{(p-1)!}\varphi_{p-1}+...+\varphi_0\right),\ p=0,...,m \tag{3}$$

satisfy the equations

$$L\left(\frac{d}{dt},\varepsilon\right)u(t)=0, \tag{4}$$

and are called elementary solutions of homogeneous equation (4).

Let's find the following vectors

$$u_p^{(\nu)}(\lambda_0,t)\Big|_{t=0}=\varphi_p^{(\nu)},\ p=1,...,m,\ \nu=0,1,2,3.$$

Definition 2. *If the system* $\left\{\left(\varphi_p^{(0)},\varphi_p^{(1)},\varphi_p^{(2)},\varphi_p^{(3)}\right)\right\}\in H^4$ *constructed by all possible eigen numbers and eigen adjoined vectors is complete in* H^4 *- sum of four copies of Hilbert space* H*, then we'll say that the system of eigen and adjoined vectors of the bundle* $L(\lambda;\varepsilon)$ *is four-fold complete in* H.

In the below-proved theorem the sufficient conditions providing four-fold completeness of system of eigen and adjoined vectors of the bundle $L(\lambda;\varepsilon)$ are established.

Theorem. *Let* $\varepsilon\in[0;1]$*, and the operators* G *and* F_j $(j=1,2,3,4)$ *satisfy condition 1)-3) and the one of the conditions cited below be satisfied:*

a) $F_jG^{-1}\in\sigma_p$, $0<p\leq 1$ *and the inequality*

$$K(\varepsilon)=\sum_{j=0}^{3}d_j(\varepsilon)\left\|F_{4-j}G^{-(4-j)}\right\|<1,$$

where

$$d_0(\varepsilon)=1,$$

$$d_1(\varepsilon)=d_3(\varepsilon)=\frac{3\sqrt{3}\left(-\varepsilon+\sqrt{\varepsilon^2+3}\right)^{1/2}}{4\left(3+\varepsilon\sqrt{\varepsilon^2+3}-\varepsilon^2\right)},$$

$$d_2(\varepsilon) = \frac{1}{2(1+\varepsilon)};$$

hold;

b) $G^{-1} \in \sigma_p \ \ (0 < p < \infty)$, $F_j G^{-j} \quad (j = 1,2,3,4)$ *are completely continuous operators.*

Then the system of eigen and adjoined vectors of $L(\lambda;\varepsilon)$ *at each* $\varepsilon \in [0;1]$ *is four-fold complete in* H.

Proof. It is obvious that the operator bundle $L_0(\lambda;\varepsilon;G)$ is reversible on imaginary axis at any $\varepsilon \in [0;1]$. Then from the equality at $\lambda = i\xi$, $\xi \in R$ we have

$$L(\lambda;\varepsilon) = L_0(\lambda;\varepsilon;G) + L_1(\lambda;\varepsilon) = \left(E + L_1(\lambda;\varepsilon) L_0^{-1}(\lambda;\varepsilon;G)\right) L_0(\lambda;\varepsilon;G).$$

We obtain that at fulfilling the condition at $\lambda = i\xi$, $\xi \in R$

$$\left\| L_1(\lambda;\varepsilon) L_0^{-1}(\lambda;\varepsilon;G) \right\| < 1 \tag{5}$$

the operator bundle $L(\lambda;\varepsilon)$ is reversible and

$$L^{-1}(\lambda;\varepsilon) = L_0^{-1}(\lambda;\varepsilon;G)\left(E + L_1(\lambda;\varepsilon) L_0^{-1}(\lambda;\varepsilon;G)\right)^{-1}. \tag{6}$$

Now we find the conditions providing the fulfilment of inequality (5) at $\lambda = i\xi$. Since at $\lambda = i\xi$, $\xi \in R$

$$\left\| L_1(i\xi;\varepsilon) L_0^{-1}(i\xi;\varepsilon;G) \right\| = \left\| \sum_{j=0}^{3} (i\xi)^j F_{4-j} \left(\xi^4 E + 2\varepsilon\xi^2 G^2 + G^4\right)^{-1} \right\| \le$$

$$\le \sum_{j=0}^{3} \left\| \xi^j F_{4-j} G^{-(4-j)} G^{(4-j)} \left(\xi^4 E + 2\varepsilon\xi^2 G^2 + G^4\right)^{-1} \right\| \le \tag{7}$$

$$\le \sum_{j=0}^{3} \left\| F_{4-j} G^{-(4-j)} \right\| \left\| \xi^j G^{(4-j)} \left(\xi^4 E + 2\varepsilon\xi^2 G^2 + G^4\right)^{-1} \right\|.$$

We estimate the norms

$$\left\| \xi^j G^{(4-j)} \left(\xi^4 E + 2\varepsilon\xi^2 G^2 + G^4\right)^{-1} \right\|, \quad j = 0,1,2,3 \tag{8}$$

it is obvious that it follows from the spectral expansion of G that

$$\left\| \xi^j G^{(4-j)} \left(\xi^4 E + 2\varepsilon\xi^2 G^2 + G^4\right)^{-1} \right\| = \sup_{\mu \in \sigma(G)} \left\| \xi^j \mu^{4-j} \left(\xi^4 + 2\varepsilon\xi^2\mu^2 + \mu^4\right)^{-1} \right\| \le$$

$$\le \sup_{\mu \in \sigma(G)} \left\| (\xi/\mu)^j \left((\xi/\mu)^4 + 2\varepsilon(\xi/\mu)^2 + 1\right)^{-1} \right\| \le \sup_{\eta \ge 0} \left| \eta^j \left(\eta^4 + 2\varepsilon\eta^2 + 1\right)^{-1} \right|. \tag{9}$$

The simple computations show that

$$\sup_{\eta \ge 0} \left| \eta^j \left(\eta^4 + 2\varepsilon\eta^2 + 1\right)^{-1} \right| = d_j(\varepsilon), \quad j = 0,1,2,3,$$

where the numbers $d_j(\varepsilon)$ are defined from the conditions of the theorem.

Then from inequality (7) we obtain

$$\left\|L_1(i\xi;\varepsilon)\,L_0^{-1}(i\xi;\varepsilon;G)\right\| \le \sum_{j=0}^{3} d_j(\varepsilon)\left\|F_{4-j}G^{-(4-j)}\right\| = K(\varepsilon) < 1.$$

Allowing for this inequality, from equality (6) we'll have that on imaginary axis

$$\left\|L^{-1}(\lambda;\varepsilon)\right\| \le \frac{1}{1-K(\varepsilon)}\left\|L_0^{-1}(\lambda;\varepsilon)\right\| \le \mathrm{const}\,(1+|\lambda|)^{-4}. \tag{10}$$

It follows from the M.V.Keldysh theory that $L^{-1}(\lambda;\varepsilon)$ is represented in the form of relation of two entire functions of order no higher than p and of minimal type. Really, from the conditions of the theorem we have

$$L(\lambda;\varepsilon) = (E+K(\lambda))\left(E+F_4G^{-4}\right)G^4.$$

But G^4+F_4 is reversible, and

$$K(\lambda) = \lambda^4T_4+\lambda^3T_3+\lambda^2T_2+\lambda T_1,$$

where

$$T_4 = G^{-4}\left(E+F_4G^{-4}\right)^{-1} \in \sigma_{p/4},$$

$$T_3 = F_1G^{-4}\left(E+F_4G^{-4}\right)^{-1} = \left(F_1G^{-1}\right)G^{-3}\left(E+F_4G^{-4}\right)^{-1} \in \sigma_{p/3},$$

$$T_2 = -2\varepsilon G^{-2}+F_2G^{-4}\left(E+F_4G^{-4}\right)^{-1} =$$

$$= -2\varepsilon G^{-2}+\left(F_2G^{-2}\right)G^{-2}\left(E+F_4G^{-4}\right)^{-1} \in \sigma_{p/2},$$

$$T_1 = F_3G^{-4}\left(E+F_4G^{-4}\right)^{-1} = \left(F_3G^{-3}\right)G^{-1}\left(E+F_4G^{-4}\right)^{-1} \in \sigma_p.$$

Since $E+K(0)=E$, then by M.B.Keldysh lemma $E+K(\lambda)$ is represented in the from of relation of two entire functions of order

$$p = \max_{j=\overline{1,4}}\left(j\cdot\frac{p}{j}\right)$$

and of minimal type at order p. $L(\lambda;\varepsilon)$ has the same property, since

$$L^{-1}(\lambda;\varepsilon) = G^{-4}\left(E+F_4G^{-4}\right)^{-1}(E+K(\lambda))^{-1}.$$

Now we prove the inverse theorem. Let the system of eigen and adjoined vectors of $L(\lambda;\varepsilon)$ be fourfold incomplete in H. Then it follows from the M.B.Keldysh theory that there exist the vectors $\psi_0,\psi_1,\psi_2,\psi_3 \in H$ such that at least one of them is nonzero, and for which

$$R(\lambda) = \left(L^*\left(\overline{\lambda};\varepsilon\right)\right)^{-1}\left(\psi_0+\lambda\psi_1+\lambda^2\psi_2+\lambda^3\psi_3\right)$$

is an entire vector function.

Then for any function $g \in H$ the scalar function

$$g(\lambda) = \left(L^{-1}(\lambda) g, \psi_0 + \overline{\lambda}\psi_1 + \overline{\lambda}^2\psi_2 + \overline{\lambda}^3\psi_3 \right)$$

will be an entire function of order no higher than ρ and of minimal type at order ρ. At fulfilling condition 1) of the theorem it is obvious that on imaginary axis

$$\left\| L^{-1}(\lambda;\varepsilon) \right\| \leq c(1+|\lambda|)^{-4}.$$

Applying the Phragmen-Lindelöf theorem we'll have

$$|g(\lambda)| \leq c(1+|\lambda|)^{-1}$$

that means that $g(\lambda) \equiv 0$. Hence it follows that $\psi_0 = \psi_1 = \psi_2 = \psi_3 = 0$. This contradicts the condition that at least one of ψ_j $\left(j = \overline{0,3}\right)$ is nonzero.

The proof of the second part of the theorem follows from M.V.Keldysh theorem [1]. The theorem is proved.

Corollary. *Let conditions 1)-3) and one of the following conditions be satisfied*

a) $G^{-1} \in \sigma_p$ $(0 < p \leq 1)$ *and the equality*

$$\sum_{j=0}^{3} d_j(0) \left\| F_{4-j} G^{-(4-j)} \right\| < 1$$

holds, where $d_0(0) = 1,\ d_1(0) = d_3(0) = \dfrac{3^{3/4}}{4},\ d_2(0) = \dfrac{1}{2};$

b) $G^{-1} \in \sigma_p$ $(0 < p \leq 1)$, *the operators* $F_j G^{-j}$ $\left(j = \overline{1,4}\right)$ *are completely continuous in* H.

Then the eigen and adjoined vectors of all bundles $L(\lambda;\varepsilon)$ at $\varepsilon \in [0,1]$ are fourfold complete in H.

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