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NECESSARY OPTIMALITY CONDITIONS IN PROBLEMS OF OPTIMAL CONTROL BY THE GOURSAT SYSTEMS WITH MULTIPOINT BOUNDARY CONDITIONS

Abstract

The optimal control problem is considered for nonlinear systems of Goursat equations with multipoint boundary conditions. The correctness questions of boundary value problem are considered. The necessary optimality condition in the form of integral maximum principle is obtained.

At the investigation of friction, sorption, drying and other processes there arise the problems of optimal control, described by the Goursat-Darboux systems [1,2]. The problem of optimal control by these systems is sufficiently studied in many papers (for example [3-5]). In these papers a system of equations characterizing the processes usually is given by the local boundary conditions. But the problem of optimal control for such systems with nonlocal boundary conditions is studied little [6,10]. At studying such problems naturally we have to investigate the correctness of the constructed boundary value problem. In some particular cases the correctness of Goursat problem with multipoint boundary conditions is studied in [7-9].

In the present paper the problems of optimal control by Goursat systems with multipoint boundary conditions are considered. The theorem on the existence, uniqueness and stability of solution of boundary value problem is cited, the necessary condition is obtained in the form of linear optimal maximum principle in the constructed optimal problem.

1. Problem statement. Let the process be described by the Goursat systems

$$y_{ts} = f\left(t, s, y, y_t, y_s, u\right), \quad (t, s) \in Q, \tag{1}$$

with multipoint boundary conditions

$$\sum_{i=1}^{N} n_i y\left(t_i, s\right) = \xi\left(s\right), \ s \in [0, l]; \ \sum_{i=1}^{M} m_j y\left(t, s_j\right) = \zeta\left(t\right), \quad t \in [0, T], \tag{2}$$

where $Q = \{(t, s) : 0 \le t \le T, \ 0 \le s \le l\}, \ (t_i, s_j), \ i = \overline{1, N}, \ j = \overline{1, M}$is an arbitrary collection of points from $Q, \ y = y\left(t, s\right)$ is n-dimensional state vector; ϕ, F and f, ξ, ζ are given scalar and vector-functions, respectively; $n_i, m_j, \ i = \overline{1, N}, \ j = \overline{1, M}$ are given matrices.

Here and below all vectors non-provided with the transposition sign (even if they are written in line) assumed to be columns.

The controls $u = u\left(t, s\right)$ are chosen from the set

$$u \in U = \left\{u : u \in L_{\infty}^{r}\left(Q\right), \ u\left(t, s\right) \in V \qquad \text{a.e.} \ \ (t, s) \in Q\right\}, \tag{3}$$

where V is given non-empty set from R^r.

It is required to minimize the functional

$$J(u)=\sum_{i=1}^{K}\phi\left(y\left(t_{\tau_i},s_{r_i}\right)\right)+\int\int_{Q} F\left(t,s,y,y_t,y_s,u\right)dtds, \qquad (4)$$

where $\left(t_{\tau_k},s_{r_k}\right)\in Q,\quad k=\overline{1,K}$.

In future we'll assume that

a) the functions $F\left(t,s,y,p,q,u\right),\ f\left(t,s,y,p,q,u\right)$ and their partial derivatives $F_y,F_p,F_q,F_u\quad f_y,f_p,f_q,f_u$ are measurable by (t,s) for all $(y,p,q,u)\in R^{3n+r}$, continuous by totality of the variables $(y,p,q,u)\in R^{3n+1}$ at a.e. $(t,s)\in Q$; the function $\phi(y)$ has continuous partial derivatives ϕ_y at all $y\in R^n$;

b) the matrices $n_i,m_j,\ i=\overline{1,N},\ j=\overline{1,M}$ are permutational $n_im_j=m_jn_i,\ i=\overline{1,N},\ j=\overline{1,M}$ and $\det\left[\sum\limits_{i=1}^{N}n_i\right]\neq 0,\ \det\left[\sum\limits_{j=1}^{M}m_j\right]\neq 0$; the functions $\xi(s),\ \xi_s(s)\in L_\infty[0,l],\ \zeta(t),\zeta_i(t)\in L_\infty[0,T]$ are such that the fitting conditions

$$\sum_{i=1}^{N}n_i\zeta\left(t_i\right)=\sum_{j=1}^{M}m_j\xi\left(s_j\right)\equiv A.$$

are fulfilled.

Under the solution of problem (1)-(2) corresponding to the control $u\in U$ it is understood the function $y(t,s;u)\in L_\infty(Q)$ having the generalized derivatives $y_t(t,s),\ y_s(t,s),\ y_{ts}(t,s)\in L_\infty(Q)$ and satisfying the differential equation (1) almost everywhere and conditions (2) in classical sense.

2. On correctness of boundary value problem. We can show that the considered boundary value problem (1)-(2) is equivalent to the following system of integral equations:

$$y(t,s)=\tilde{n}^{-1}(N)\,\xi(s)+\tilde{m}^{-1}(M)\,\zeta(t)-\tilde{n}^{-1}(N)\sum_{i=1}^{N}n_i\int_0^{t_i}\int_0^{s}f\left(\tau,r,y,y_t,y_s,u\right)drd\tau-$$

$$-\tilde{m}^{-1}(M)\sum_{i=1}^{N}m_j\int_0^{t}\int_0^{s_j}f\left(\tau,r,y,y_t,y_s,u\right)drd\tau-$$

$$-\tilde{m}^{-1}(M)\,\tilde{n}^{-1}(N)\,A+\tilde{n}^{-1}(N)\,\tilde{m}^{-1}(M)\times$$

$$\times\sum_{i=1}^{N}\sum_{j=1}^{M}n_im_j\int_0^{t_i}\int_0^{s_j}f\left(\tau,r,y,y_t,y_s,u\right)drd\tau+\int_0^{t}\int_0^{s}f\left(\tau,r,y,y_t,y_s,u\right)drd\tau,\ (t,s)\in Q, \qquad (5)$$

where

$$\tilde{n}^{-1}(N)=\left[\sum_{i=1}^{N}n_i\right]^{-1},\quad \tilde{m}^{-1}(M)=\left[\sum_{j=1}^{M}m_j\right]^{-1}$$

Let the following conditions be fulfilled:

1. $|f(t,s,0,0,0,u)| \le M_0,\ |f_y(t,s,y,p,q,u)| \le M_1,$
$|f_p(t,s,y,p,q,u)| \le M_2,\ |f_q(t,s,y,p,q,u)| \le M_3,$
a.e. $(t,s) \in Q$ for all $(y,p,q,u) \in R^{3n} \times V$, where $M_i \ge 0,\ i=\overline{0,3}$ are constants.

2. $|f_u(t,s,y,p,q,u)| \le M,$
a.e. $(t,s) \in Q$ for all $(y,p,q,u) \in R^{3n} \times V$ where $M \ge 0$ are constants.

3. $|F(t,s,0,0,0,u)|,\ |F_y(t,s,y,p,q,u)|,\ |F_p(t,s,y,p,q,u)|,\ |F_q(t,s,y,p,q,u)|,$
$|F_u(t,s,y,p,q,u)|,\ |\phi_y(y)| \le M$ and the functions $F_y, F_p, F_q, F_u, f_y, f_p, f_q, f_u, \phi_y$ satisfy the Lipchitz condition by (y,p,q,u).

Theorem 1. *Let conditions 1 be fulfilled and besides*

$$\sum_{i=1}^{3} r_i M_i < 1,$$

where

$$r_1 = T\left|\tilde{m}^{-1}(M)\right|\sum_{j=1}^{M} m_j s_j + l\left|\tilde{n}^{-1}(N)\right|\sum_{i=1}^{N} n_i t_i +$$

$$+Tl\left|\tilde{m}^{-1}(M)\right|\left|\tilde{n}^{-1}(N)\right|\sum_{i=1}^{N} n_i t_i \sum_{j=1}^{M} m_j s_j + Tl,$$

$$r_2 = \left|\tilde{m}^{-1}(M)\right|\sum_{j=1}^{M} m_j s_j + l,\ r_3 = \left|\tilde{n}^{-1}(N)\right|\sum_{i=1}^{N} n_i t_i + T.$$

Then at every fixed admissible control problem (1), (2) has a unique solution and the following estimations

$$\max_{Q}|y(t,s)|,\ \max_{[0,l]} \operatorname*{ess\,sup}_{[0,T]} |y_t(t,s)|,\ \max_{[0,T]} \operatorname*{ess\,sup}_{[0,l]} |y_s(t,s)| \le C,$$

are true, where C is a constant not depending on control.

The proof of the theorem is carried out by the method of successive approximations by the formula

$$y^{n+1} = \tilde{n}^{-1}(N)\,\xi(s) + \tilde{m}^{-1}(M)\,\zeta(t) -$$

$$-\tilde{n}^{-1}(N)\sum_{i=1}^{N} n_i \int_0^{t_i}\int_0^{s} f\left(\tau,r,y^n,y_t^n,y_s^n,u\right) d\tau dr -$$

$$-\tilde{m}^{-1}(M)\sum_{i=1}^{M} m_j \int_0^{t}\int_0^{s_j} f\left(\tau,r,y^n,y_t^n,y_s^n,u\right) dr d\tau - \tilde{m}^{-1}(M)\,\tilde{n}^{-1}(N)\,A +$$

$$+\tilde{n}^{-1}(N)\,\tilde{m}^{-1}(M)\sum_{i=1}^{N}\sum_{i=1}^{M} n_i m_j \int_0^{t_i}\int_0^{s_j} f\left(\tau,r,y^n,y_t^n,y_s^n,u\right) dr d\tau +$$

$$+\int_0^{t}\int_0^{s} f\left(\tau,r,y^n,y_t^n,y_s^n,u\right) dr d\tau, \qquad n = 0,1,2,...$$

Lemma. *Let the conditions of theorem 1 and condition 2 be fulfilled. Besides,* $u(t,s)$, $u(t,s)+u(t,s)$ *be two admissible controls, and* $y(t,s)$, $y(t,s)+y(t,s)$ *be solutions of problems (1), (2) corresponding to them. Then the following estimations are true:*

$$\max_{Q}|\bar{y}(t,s)|,\ \max_{[0,l]} ess\sup_{[0,T]}|\bar{y}_t(t,s)|,\ \max_{[0,T]} ess\sup_{[0,l]}|\bar{y}_s(t,s)| \le C\|\bar{u}\|_{L_\infty}, \qquad (6)$$

where C is a constant not depending on control.

3. Necessary optimality condition

Consider the following system of equations:

$$\psi(t,s)=\sum_{k=1}^{K}\Bigg(\tilde{n}^{-1}(N)\sum_{i=1}^{N}n_i\delta_i(t)\,\delta_{r_k}(s)+\tilde{m}^{-1}(M)\sum_{j=1}^{M}m_i\delta_i(s)\,\delta_{\tau_k}(t)-$$

$$-\tilde{n}^{-1}(N)\,\tilde{m}^{-1}(M)\sum_{i=1}^{N}\sum_{j=1}^{M}n_im_j\delta_i(t)\,\delta_j(s)-\delta_{\tau_k}(t)\,\delta_{r_k}(s)\,E\Bigg)'\Phi_y\left(y\left(t_{\tau_k},s_{r_k}\right)\right)+$$

$$+\tilde{n}^{-1}(N)\,\tilde{m}^{-1}(M)\sum_{i=1}^{N}\sum_{j=1}^{M}n_im_j\delta_i(t)\,\delta_j(s)\int_0^T\int_0^l\tilde{H}_y(t,s)\,dtds+\int_0^t\int_0^s\tilde{H}_y(t,s)\,d\tau dr-$$

$$-\int_0^s\tilde{H}_p(t,r)\,dr-\int_0^t\tilde{H}_q(\tau,s)\,d\tau+$$

$$+\left(\tilde{n}^{-1}(N)\sum_{i=1}^{N}n_i\delta_i(t)\right)'\left(\int_0^T\tilde{H}_q(t,s)\,dt-\int_0^T\int_0^s\tilde{H}_y(t,r)\,drdt\right)+$$

$$+\left(\tilde{m}^{-1}(M)\sum_{j=1}^{M}m_j\delta_j(s)\right)'\left(\int_0^l\tilde{H}_p(t,s)\,ds-\int_0^t\int_0^l\tilde{H}_y(\tau,s)\,d\tau ds\right),\ (t,s)\in Q,\ (7)$$

where

$$\delta_i(t)=\begin{cases}0, & 0\le t<t_i\\ 1, & t_i\le t\le T,\end{cases}\qquad \delta_i(s)=\begin{cases}0, & 0\le s<s_i\\ 1, & s_i\le s\le l,\end{cases}$$

$$H(t,s,y,y_t,y_s,u,\psi)=-F(t,s,y,y_t,y_s,u)+<\psi,\ f(t,s,y,y_t,y_s,u)>,$$

$\tilde{H}(t,s)=H(t,s,y,y_t,y_s,u,\psi)$, E is a unique matrix of order $n\times n$. (7) is called an adjoint system.

Theorem 2. *Let conditions 1-3 be fulfilled and U be convex. Then for optimality of control $u(t,s)\in U$ it is necessary the fulfilment of the inequality*

$$\int_0^T\int_0^l\langle H_u(t,s),\nu(t,s)-u(t,s)\rangle\,dtds\le 0,$$

for all $\nu(t,s) \in U$.

Proof. Let $y, y+\bar{y}$ be a solution of problem (1), (2) corresponding to the controls $u, u+u \in U$. Then from (1), (2) we obtain

$$y_{ts} = f\left(t, s, y+y,\ y_t+y_t,\ y_s+y_s,\ u+u\right) - f\left(t, s, y, y_t, y_s, u\right),\ (t,s) \in Q, \tag{8}$$

$$\sum_{i=1}^{N} n_i \bar{y}\left(t_i, s\right) = 0,\ s \in [0,l]\,; \sum_{j=1}^{M} m_j \bar{y}\left(t, s_j\right) = 0,\ t \in [0,T]\,, \tag{9}$$

Calculating the increment of functional (4) we'll obtain

$$\Delta J(u) = J(u+\bar{u}) - J(u) = \sum_{i=1}^{K}\left[\phi\left(y\left(t_i,s_i\right)+\bar{y}\left(t_i,s_i\right)\right) - \phi\left(y\left(t_i,s_i\right)\right)\right] +$$

$$+\int_0^T\int_0^l \left[F\left(t,s,y+\bar{y},(y+\bar{y})_t,(y+\bar{y})_s,u+\bar{u}\right) - F\left(t,s,y,y_t,y_s,u\right)\right]dtds. \tag{10}$$

Multiply equality (8) by some function $\psi(t,s)$ integrate by the domain Q and add to (10)

$$\Delta J(u) = \sum_{i=1}^{K}\left\langle \Phi_y\left(y\left(t_{\tau_i}, s_{r_i}\right), z\left(t_{\tau_i}, s_{r_i}\right)\right)\right\rangle -$$

$$-\int_o^T\int_o^l \left[\left\langle \tilde{H}_y(t,s), \bar{y}(t,s)\right\rangle + \left\langle \tilde{H}_p(t,s), \bar{y}_i(t,s)\right\rangle + \right.$$

$$\left. + \left\langle \tilde{H}_q(t,s), \bar{y}_s(t,s)\right\rangle + \left\langle \tilde{H}_u(t,s), \bar{u}(t,s)\right\rangle - \left\langle \psi, \bar{y}_{ts}\right\rangle\right] dtds + \eta, \tag{11}$$

$$\eta = \sum_{i=1}^{K} o_i\left(\left|\bar{y}\left(t_{\tau_i}, s_{r_i}\right)\right|\right) - \int_0^T\int_0^l o_{K+1}\left(\left|\bar{y}(t,s)\right| + \left|\bar{y}(t,s)\right| + \left|\bar{y}_s(t,s)\right| + \left|\bar{u}(t,s)\right|\right) dtds.$$

Using the integration by parts formula and by virtue of Fubini theorem we have

$$\int_o^T\int_o^l \left\langle \tilde{H}_y(t,s), \bar{y}(t,s)\right\rangle dtds = \int_o^T\int_o^l \left\langle \tilde{H}_y(t,s)\,dtds, \bar{y}(T,l)\right\rangle -$$

$$-\int_o^T\int_o^l < \int_o^s \tilde{H}_y(t,s)\,dr, \bar{y}_s(T,s) > dtds -$$

$$-\int_o^T\int_o^l < \int_o^t \tilde{H}_y(\tau,s)\,d\tau, \bar{y}_s(t,l) > dtds +$$

$$+\int\limits_o^T\int\limits_o^l<\int\limits_o^T\int\limits_o^s\tilde{H}_y(\tau,r)\,d\tau dr,\ \bar{y}_{ts}(t,s)>dtds, \tag{12}$$

$$-\int\limits_o^T\int\limits_o^l<\int\limits_o^t\tilde{H}_p(t,s),\bar{y}_t(t,s)>dtds=-\int\limits_o^T\int\limits_o^l<\tilde{H}_p(t,s),\bar{y}_s(t,l)>dtds-$$

$$-\int\limits_o^T\int\limits_o^l<\int\limits_o^s\tilde{H}_p(t,r)\,dr,\bar{y}_{ts}(t,s)>dtds, \tag{13}$$

$$\int\limits_o^Y\int\limits_o^l\left\langle\tilde{H}_q(t,s),\bar{y}_s(t,s)\right\rangle dtds=\int\limits_o^T\int\limits_o^l\left\langle\tilde{H}_q(t,s),\bar{y}_s(T,s)\right\rangle dtds-$$

$$-\int\limits_o^T\int\limits_o^l\left\langle\int\limits_o^l\tilde{H}_q(\tau,s)\,d\tau,\bar{y}_{ts}(t,s)\right\rangle dtds. \tag{14}$$

Let's use the identity

$$\bar{y}(t_i,s_i)=\bar{y}(T,l)-\int\limits_o^T\bar{y}_t(t,l)\,\delta_i(t)\,dt-$$

$$-\int\limits_o^l\bar{y}_s(T,s)\,\delta_i(s)\,ds+\int\limits_o^T\int\limits_o^l\bar{y}_{ts}(\tau,s)\,\delta_i(t)\,\delta_i(s)\,dtds. \tag{15}$$

Allowing for equalities (12)-(15) in (11) we'll obtain

$$\Delta J(u)=<\sum_{i=1}^K\Phi_y\left(y\left(t_{\tau_k},s_{r_k}\right)\right)-\int\limits_o^T\int\limits_o^l\tilde{H}_y(t,s)\,dtds,\bar{y}(T,l)>+$$

$$+\int\limits_o^l<\int\limits_o^T\int\limits_o^l\tilde{H}_y(t,r)\,drdt-\int\limits_o^T\tilde{H}_q(t,s)\,dt-$$

$$-\sum_{i=1}^K\Phi_y\left(y\left(t_{\tau_k},s_{r_k}\right)\right)\delta_{r_k}(s),y_s(T,s)>ds+\int\limits_o^T\int\limits_o^t\int\limits_o^l\tilde{H}_y(\tau,s)\,d\tau ds-$$

$$-\int\limits_o^l\tilde{H}_p(t,s)\,ds-\sum_{i=1}^K\Phi_y\left(y\left(t_{\tau_k},s_{r_k}\right)\right)\times$$

$$\times\delta_{\tau_k}(t),y_t(t,l)>dt+\int\limits_o^T\int\limits_o^l<\left[\psi-\int\limits_o^t\int\limits_o^s\tilde{H}_y(t,s)\,d\tau dr+\right.$$

$$+\int\limits_o^s \tilde{H}_p(t,r)\,dr+\int\limits_o^s \tilde{H}_q(\tau,s)\,d\tau+$$

$$+\sum_{i=1}^{K}\Phi_y\left(y\left(t_{\tau_k},s_{r_k}\right)\right)\delta_{r_k}(t)\,,\delta_{r_k}(s)\Bigg],\bar{y}_{ts}(t,s)>$$

$$>dtds-\int\limits_o^T\int\limits_o^l<\tilde{H}_u(t,s)\,,\bar{u}>dtds+\eta. \tag{16}$$

We can write the first condition (9) in the form

$$\sum_{i=1}^{N}n_i\left(\bar{y}(T,s)-\int\limits_{t_i}^{T}\bar{y}_t(t,s)\,dt\right)=0,\quad s\in[0,l]\,.$$

Hence we'll find

$$\bar{y}(T,s)=\tilde{n}^{-1}(N)\sum_{i=1}^{N}n_i\int\limits_0^T\bar{y}_t(t,s)\,\delta_i(t)\,dt,\quad s\in[0,l]\,,$$

$$\bar{y}_s(T,s)=\tilde{n}^{-1}(N)\sum_{i=1}^{N}n_i\int\limits_0^T\bar{y}_{ts}(t,s)\,\delta_i(t)\,dt,\quad s\in[0,l]\,. \tag{17}$$

From the second condition (9) we have

$$\bar{y}(t,l)=\tilde{m}^{-1}(M)\sum_{j=1}^{M}m_j\int\limits_0^l\bar{y}_s(t,s)\,\delta_j(s)\,ds,\quad t\in[0,T]\,, \tag{18}$$

$$\bar{y}_t(t,l)=\tilde{m}^{-1}(M)\sum_{j=1}^{M}m_j\int\limits_0^l\bar{y}_{ts}(t,s)\,\delta_j(s)\,ds,\quad t\in[0,T]\,, \tag{19}$$

analogously.

From (17) and (18) we obtain that

$$\bar{y}(T,l)=\tilde{n}^{-1}(N)\,\tilde{m}^{-1}(M)\sum_{i=1}^{N}\sum_{j=1}^{M}n_im_j\int\limits_0^T\int\limits_0^l\bar{y}_{ts}(t,s)\,\delta_i(t)\,\delta_j(s)\,dtds. \tag{20}$$

Allowing for equality (17), (19), (20) in equality (16) and choosing $\psi(t,s)$ as a solution of system (7) we have

$$\Delta J(u)=-\int\limits_0^T\int\limits_0^l<\tilde{H}_u(t,s)\,,\bar{u}(t,s)>dtds+\eta. \tag{21}$$

Let $u(t,s)$ be optimal control in problem (1)-(2). Then taking in (21) $\bar{u}(t,s;\varepsilon)=\varepsilon(\nu(t,s)-u(t,s))$, where $0\le\varepsilon\le 1$, $\nu(t,s)\in U$, $(t,s)\in Q$ is any control and from estimation (6) we have

$$\Delta J(u)=-\varepsilon\int_0^T\int_0^l<\tilde{H}_u(t,s),\nu(t,s)-u(t,s)>dtds+o(\varepsilon)\ge 0,$$

hence it follows the truth of the theorem.

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