APPLIED PROBLEMS OF MATHEMATICS AND MECHANICS

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THE OPTIMAL CONTROL PROBLEM FOR GOURSAT-DARBOUX LINEAR SYSTEM

Abstract

In the given work the optimal control problem in linear hyperbolic system is investigated by method of l -problem of moments.

By investigating various processes, such as sorption, drying and etc., optimal control problems, described by a system of hyperbolic equations [2, 4, 8, 9] arise.

Let the controlled process be described by the equation:

$$x_{ts} = A_1(t,s)\, x_t + A_2(t,s)\, x_s + A_3(t,s)\, x + c(t,s)\, \omega(t,s) \in D \tag{1}$$

with conditions

$$x_t(t,s_0) = B_1(t)\, x(t,s_0) + c_1(t)\, u,\ \ t_0 \le t \le t_1, \tag{2}$$

$$x_s(t_0,s) = B_2(s)\, x(t_0,s) + c_2(s)\, v,\ \ s_0 \le s \le s_1, \tag{3}$$

$$x(t_0,s_0) = x^0, \tag{4}$$

where $A_i(t,s)\,(i=1,2,3)$, $B_1(t)$, $B_2(s)$ are $n\times n$ matrices, $c(t,s)$ is $n\times m$ matrix, $c_1(t)$ is $n\times m_1$ matrix, $c_2(s)$ is $n\times m_2$ matrix, $(\omega(t,s)\ \ u(t),\ v(s))$ is $m+m_1+m_2$ dimensional vector control, $D=[t_0,t_1]\times[s_0,s_1]$.

In future, as feasible controls $(\omega(t,s),\ u(t), v(s))$ we'll consider the elements of the space $W_p(D) = L_p^m(D)\times L_p^{m_1}[t_0,t_1]\times L_p^{m_2}[s_0,s_1]$, where $L_p^m(D)$ is a space of such measurable on D m-dimensional vector-functions $\omega(t,s)$, that

$$\|\omega\|_p = \left(\sum_{i=1}^{m}\iint_D \left|\omega^i(t,s)\right|^p dsdt\right)^{1/p} < \infty \ \text{ as } 1\le p<\infty,$$

and

$$\|\omega\|_\infty = \max_{1\le i\le m}\left\{vrai\max_D\left|\omega^i(t,s)\right|\right\} < \infty \ \text{ for }\ p=\infty.$$

Let us note, that the space $W_p(D)$ is a direct product of three Banach spaces $L_p^m(D)$, $L_p^{m_1}[t_0,t_1]$, $L_p^{m_2}[s_0,s_1]$. This space is also a Banach space with norm [10, page 47]

$$\|(\omega,u,v)\|_{W_p(D)} = \|\omega\|_{L_p^m(D)} + \|u\|_{L_p^{m_1}[t_0,t_1]} + \|v\|_{L_p^{m_2}[s_0,s_1]}.$$

The space $W_p(D)$ is adjoint to the space $W_q(D) = L_q^m(D)\times L_q^{m_1}[t_0,t_1]\times L_q^{m_2}[s_0,s_1]$ with norm [10, pp.47-48]

$$\|(\gamma,\alpha,\beta)\|_{W_q(D)} = \max\left\{\|\gamma\|_{L_q^m(D)}, \|\alpha\|_{L_q^{m_1}[t_0,t_1]}, \|\beta\|_{L_q^{m_2}[s_0,s_1]}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1,\ (\gamma,\alpha,\beta)\in W_q(D)$.

The duality between $W_q(D)$ and $W_p(D)$ is defined by the functional

$$L[(\gamma,\alpha,\beta)]=\iint_D \gamma(t,s)\,\omega(t,s)\,dsdt+\int_{t_0}^{t_1}\alpha(t)\,u(t)\,dt+\int_{s_0}^{s_1}\beta(s)\,\upsilon(s)\,ds. \tag{5}$$

At that $\|L\|=\|(\omega,u,\upsilon)\|_{W_p(D)}$.

At the given control $(\omega(t,s),\ u(t),\ \upsilon(s))\in W_p(D)$ the solution of problem (1)-(4) is n -dimensional vector-fucntion $x(t,s)$, absolutely continuous [7, p.246] and satisfying equations (1)-(3) and condition (4) almost everywhere.

Suppose, that the following conditions are fulfilled:

1. The matrices $A_i(t,s)$, $\frac{\partial A_i}{\partial t}$, $A_2(t,s)$, $\frac{\partial A_2}{\partial s}$, $A_3(t,s)$, $B_1(t)$, $B_2(s)$ are measurable and their norms are integrated by Lebesgue,

2. The rows $(c(t,s),c_1(t),c_2(s))^j$ $(j=1,2,...,n)$ of the matrix $(c(t,s),c_1(t),c_2(s))$ belong to the space $W_q(D)$.

Under indicated conditions, the existence and uniqueness of the solutions of problem (1)-(4) can be proved by the ordinary successive approximations method [8, 9].

For finding the correlation, expressing the solution of system (1)-(4), we'll introduce $n\times n$-matrices $\Phi_0(t,s)$ $(t_0\le t\le t_1,\ s_0\le s\le s_1)$, $\Phi_1(t,s,\tau)$, $(t_0\le t\le t_1,$ $s_0\le s\le s_1,\ t_0\le\tau\le t)$, $\Phi_2(t,s;\sigma)$ $(t_0\le t\le t_1, s_0\le s\le s_1,\ s_0\le\sigma\le s)$, $\Phi_3(t,s;\tau,\sigma)$ $(t_0\le t\le t,\ s_0\le s\le s_1,\ t_0\le\tau\le t,\ s_0\le\tau\le s)$, defined as solution of the system

$$\frac{\partial^2\Phi}{\partial t\partial s}=A_1(t,s)\frac{\partial\Phi}{\partial t}+A_2(t,s)\frac{\partial\Phi}{\partial s}+A_3(t,s)\,\Phi \tag{6}$$

satisfying the conditions

$$\frac{\partial\Phi_0(t,s_0)}{\partial t}=B_1(t)\,\Phi_0(t,s_0),\ \frac{\partial\Phi_0(t_0,s)}{\partial s}=B_2(s)\,\Phi_0(t_0,s),$$

$$\Phi_0(t_0,s_0)=I;\ \frac{\partial\Phi_1(t,s;t)}{\partial s}=A_1(t,s)\,\Phi_1(t,s,t), \tag{7}$$

$$\frac{\partial\Phi_1(t,s_0;\tau)}{\partial t}=B_1(t)\,\Phi_1(t,s_0;\tau),\Phi_1(t,s_0,t)=I,$$

$$\frac{\partial\Phi_2(t,s;s)}{\partial t}=A_2(t,s)\,\Phi_2(t,s;s),\frac{\partial\Phi_2(t_0,s;\sigma)}{\partial s}=B_2(s)\,\Phi_2(t_0,s;\sigma),$$

$$\Phi_2(t,s;s)=I,\ \frac{\partial\Phi_3(t,s;\tau,s)}{\partial t}=A_2(t,s)\,\Phi_3(t,s;\tau,s),$$

$$\frac{\partial\Phi_3(t,s;t,\sigma)}{\partial s}=A_1(t,s)\,\Phi_3(t,s;t,\sigma),\Phi_3(t,s;t,s)=I.$$

I is a unique $n\times n$ -matrix.

Let $x(t,s)$ be a unique absolutely continuous solution of problem (1)-(4) at the control $(\omega(t,s), u(t), \upsilon(s)) \in W_p(D)$. Then $x(t,s)$ is represented by the formula

$$x(t,s) = \Phi_0(t,s)x^0 + \int_{t_0}^{t} \Phi_1(t,s;\tau) c_1(\tau) u(\tau) d\tau +$$

$$+\int_{s_0}^{s} \Phi_2(t,s;\sigma) c_2(\sigma) \upsilon(\sigma) d\sigma + \int_{t_0}^{t}\int_{s_0}^{t} \Phi_3(t,s;\tau,\sigma) c(\tau,\sigma) \omega(\tau,\sigma) d\sigma d\tau, \qquad (8)$$

where $\Phi_0(t,s), \Phi_1(t,s;\tau), \Phi_2(t,s;\sigma), \Phi_3(t,s;\tau,\sigma)$ are defined from conditions (6), (7).

The following problem is stated: it is required to find such a control $(\omega(t,s), u(t), \upsilon(s)) \in W_p(D)$, that its corresponding solution $x(t,s)$ of system (1)-(4) satisfies the condition

$$x(t_1,s_1) = x^* \qquad (9)$$

and has, at that, the least norm.

$$\|(\omega,u,\upsilon)\|_{W_p(D)},$$

where $x^* \in R$ is the given point.

Let the solution $x(t,s)$ of problem (1)-(4) at the control $(\omega(t,s),\ u(t),\ \upsilon(s))$ satisfy condition (9). Then from (8) we obtain, that for such a control the equality

$$\iint_D f(t,s)\omega(t,s) ds dt + \int_{t_0}^{t_1} g(t) u(t) dt + \int_{s_0}^{s_1} h(s) \upsilon(s) ds = a \qquad (10)$$

where

$$a = x^* - \Phi_0(t_1,s_1)x^0, f(t,s) = \Phi_3(t_1,s_1;t,s)c(t,s),$$

$$g(t) = \Phi_1(t_1,s_1;t)c_1(t),\ h(s) = \Phi_2(t_1,s_1;s)c_2(s) \qquad (11)$$

is fulfilled.

The fulfilment of equality (10) is the necessary and sufficient condition in order that the solution $x(t,s)$ of problem (1)-(4), corresponding to the control $(\omega(t,s), u(t), \upsilon(s))$ satisfy condition (9). Equality (10) expresses the problem of moments, noted in vector-matrix form [1, 3, 5, 6]. Using equalities (5) and (10) the problem of l -problem of moments it is possible to formulate in the following form. It is required to find such a linear functional L, defined on elements of the space $W(D)$, that

$$L[(f,g,h)^j] = a^j,\ j = 1,2,...,n, \qquad (12)$$

$$\|L\| \le l\ \ (l > 0), \qquad (13)$$

be fulfilled, where $(f,g,h)^j$ is the j -th row of the matrix $f(t,s), g(t), h(s)$ and a^j is the j -th component of the vector a.

Taking into account, $\|L\| = \|(\omega, u, v)\|_{W_p(D)}$, we obtain, that for solution of the stated problem it is required to find the linear functional L, which is a solution of l -problem of moments (12), (13), at that l is the least number.

Lemma. *Let the rows $(f,g,h)^j$ $(j = 1,2,3,...,n)$ of the matrix $(f(t,s)$, $g(t)$, $h(s))$ be linear independent and $a \neq 0$.*

Then the problem

$$\lambda = \max_{\xi} \xi a, \tag{14}$$

$$\|\xi(f,g,h)\|_{W_q(D)} = 1, \tag{15}$$

has a solution, where ξ is an n -dimensional vector-row.

Proof. First of all let us note, that the dual to problem (14), (15) is the following equivalent problem (see [6])

$$\frac{1}{\lambda} = \min_{\eta} \|\eta(f,g,h)\|_{W_q(D)} \tag{16}$$

provided

$$\eta a = 1. \tag{17}$$

Under the conditions of lemma the set $\eta(f(t,s), g(t), h(s))$ forms n -dimensional space E_n, stretched on the rows of the matrix $(f(t,s), g(t), h(s))$, where η is any n -dimensional vector-row. In the space E_n the norm is defined as in the space $W_q(D)$.

Let $\{\eta^k\}$ be a minimizing sequence of vector-rows, i.e.

$$\frac{1}{\lambda} = \min_{\eta} \|\eta(f,g,h)\|_{W_q(D)} = \lim_{k\to\infty} \left\|\eta^k(f,g,h)\right\|_{W_q(D)},$$

$$\eta a = 1,\ \eta^k a = 1,\ k = 1,2,....$$

Hence it follows, that the sequence $\{\eta^k(f,g,h)\}$ is uniformly bounded in E_n.

Therefore, from this sequence it is possible to choose such a sequence (we'll again define it by $\{\eta^k(f,g,h)\}$), for which the limit exists:

$$\lim_{k\to\infty} \eta^k(f,g,h) = \eta^\circ(f,g,h).$$

Hence, we have

$$\|\eta^\circ(f,g,h)\|_{W_q(D)} = \lim_{k\to\infty} \left\|\eta^k(f,g,h)\right\|_{W_q(D)} = \frac{1}{\lambda},$$

$$\eta^\circ = \lim_{k\to\infty} \eta^k, \eta^\circ a = 1.$$

The lemma is proved.

Theorem 1. *Let the conditions of the lemma be fulfilled. Then for the existence of the solution of problem (12), (13) it is necessary and sufficient, that the condition $\lambda \leq l$, where λ is a solution of problem (14), (15), be fulfilled.*

Proof of necessity. Let there exist the linear functional L, defined on elements of the space $W_q(D)$, which will give the solution of the problem of moments (12), (13).

From equalities (12) we have

$$L[\eta\,(f,g,h)] = \eta a.$$

From here, taking into account inequality (13) we obtain

$$|\eta a| = |L[\eta\,(f,g,h)]| \le \|L\| \;\; \|\eta\,(f,g,h)\|_{W_q(D)} \le$$
$$\le l\,\|\eta\,(f,g,h)\|_{W_q(D)}\,.$$

With regard to (17) we have

$$\|\eta\,(f,g,h)\|_{W_q(D)} \ge \frac{1}{l}.$$

Consequently, and minimal value $\|\eta\,(f,g,h)\|_{W_q(D)}$ under all η, satisfying equality (17), is no less than $1/l$, i.e.

$$\min_{\eta} \|\eta\,(f,g,h)\|_{W_q(D)} = \frac{1}{\lambda} \ge \frac{1}{l}, \quad \eta a = 1.$$

From here, $l \ge \lambda$. The necessity is proved. To prove the sufficiency we'll assume, that the condition $l \ge \lambda$ is fulfilled. Let's define on n dimensional linear space E_n the linear functional L_0 in the following form $L_0[\eta\,(f,g,h)] = \eta a$ with the norm $\|L_0\| = \lambda$, where λ is a solution of problem (14), (15).

By Khan-Banach theorem on extension of linear functional, with norm preservation, there exists the linear functional L defined on the elements of the space $W_q\,(D)$, such that $L[(\gamma,\alpha,\beta)] = L_0[(\gamma,\alpha,\beta)]$ for $(\gamma,\alpha,\beta) \in E_n$ and $\|L\| = \|L_0\| = \lambda$ (see: [10], p.97).

The linear functional L, defined on the space $W_q\,(D)$ has form (5) and $\|L\| = \|(\omega,u,v)\|_{W_p(D)}$.

Thus, L is the required functional, which gives us the solution of problem (12), (13).

The theorem is proved.

Theorem 2. *Under the lemma's condition, there exists the solution of problem (1)-(4), (9).*

Proof. Let the number λ be defined as a solution of problem (14), (15). By Theorem 1 there exists the control $(\omega\,(t,s)\,,u\,(t)\,,v\,(s)) \in W_p\,(D)$ such, that for the functional $L[(\gamma,\alpha,\beta)] = \int\!\!\int_D \gamma\,(t,s)\,\omega\,(t,s)\,dsdt + \int_{t_0}^{t_1} \alpha\,(t)\,u\,(t)\,dt + \int_{s_0}^{s_1} \beta\,(s)\,v\,(s)\,ds$ equality (10) is fulfilled and $\|L\| = \|(\omega,u,v)\|_{W_p(D)} = \lambda$.

From here, it follows, that $(\omega\,(t,s)\,,u\,(t)\,,v\,(s)) \in W_p\,(D)$ is an optimal control. The theorem is proved.

Let's suppose, that $1 < p < \infty$ and consider the sequence of solutions of the optimization problem by method of moments:

1. Define the number λ^* and the vector ξ^* as a solution of the problem

$$\lambda^* = \xi^* a = \max_{\xi} \xi a \tag{18}$$

provided

$$\|\xi^*\,(f,g,h)\|_{W_q(D)} = 1, \tag{19}$$

where $(f(t,s),\ g(t),\ h(s))$ is defined by formula (11),

$$\|\xi^*(f,g,h)\|_{W_q(D)} = \max\left\{\|\xi^* f\|_{L_q^m(D)},\ \|\xi^* g\|_{L_q^m(t_0,t_1)},\ \|\xi^* h\|_{L_q^{m_2}[s_0,s_1]}\right\}.$$

2. From condition (19) it follows, that at least one of the equalities

$$\|\xi^* f\|_{L_q^m(D)} = 1,\ \|\xi^* g\|_{L_q^{m_1}[t_0,t_1]} = 1,\ \|\xi^* h\|_{L_q^{m_2}[s_0,s_1]} = 1$$

is fulfilled.

If just the first equality is fulfilled, then the optimal control $(\omega^*(t,s),\ u^*(t), v^*(s)) \in W_p(D)$ is defined in the following form:

$$\omega_k^*(t,s) = \lambda^* \left|(\xi^* f(t,s))^k\right|^{q-1} sign\,(\xi^* f(t,s))^k, \quad k = 1,2,...m,$$

$u^*(t) = 0,\ v^*(s) = 0$ where $(\xi^* f(t,s))^k$ is the k -th component of the vector $\xi^* f(t,s)$.

At that

$$\|(\omega^*, u^*, v^*)\|_{W_p(D)} = \lambda^*.$$

Other cases is analyzed in a similar way

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Received March 01, 2003; Revised May 26, 2004.
Translated by Mamedova Sh.N.