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L^q-MEAN LIMITS FOR TAYLOR EXPANSIONS OF ANISOTROPIC RIESZ POTENTIALS

Abstract

In this paper L^q-mean limits for Taylor expansions of anisotropic Riesz potentials are studied and the theorem on L^q-differentiability of anisotropic Riesz potentials is given. The obtained results are generalizations of corresponding T.Shimomura's results for the classical Riesz potentials (see [10]).

Introduction. For $x = (x_1, x_2, ..., x_n)$ we define the following $\lambda-$distance from zero

$$\|x\|_\lambda = \left(\sum_{i=1}^{n} |x_i|^{\frac{1}{\lambda_i}} \right)^{\frac{|\lambda|}{n}},$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are positive numbers and $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$ (see [1]).

It is clear that for any $t > 0$

$$\left\| t^\lambda x \right\|_\lambda = t^{\frac{|\lambda|}{n}} \|x\|_\lambda .$$

Taking the notation $\underline{\lambda} = \min\limits_{i=\overline{1,n}} \lambda_i$, $\overline{\lambda} = \max\limits_{i=\overline{1,n}} \lambda_i$, we can note that triangle inequality for λ-distance will have the form

$$\|x + y\|_\lambda \leq 2^{\frac{|\lambda|}{\underline{\lambda} n}} \left(\|x\|_\lambda + \|y\|_\lambda \right).$$

Denote by $\sigma(x_0, r)$ an open ball of radius r with the center x_0

$$\sigma(x_0, r) = \{ x \in R^n : \|x - x_0\|_\lambda < r \}.$$

For $0 < \alpha < n$ the function $R_\alpha(y) = \|y\|_\lambda^{\alpha - n}$ is called an anisotropic Riesz kernel, and the integral

$$R_\alpha f(x) = \int\limits_{R^n} \|x - y\|_\lambda^{\alpha - n} f(y)\, dy \tag{1}$$

is called an anisotropic Riesz potential.

For the existence of the potential $R_\alpha f$ almost everywhere, in the case of nonnegative locally-integrable function f it is necessary and sufficient that

$$\int\limits_{R^n} \left(1 + \|y\|_\lambda\right)^{\alpha - n} f(y)\, dy \tag{2}$$

(see [3] theorem 1, corollary 1 and 2).

Let $q > 0$, $x_0 \in R^n$, $r > 0$ and u be a measurable function in $\sigma(x_0, r)$. Then the quantity

$$V_q(u, x_0, r) = \left(\frac{1}{m(\sigma(x_0, r))} \int\limits_{\sigma(x_0, r)} |u(x)|^q\, dx \right)^{\frac{1}{q}}$$

is called L^q-mean value of the function u in the ball $\sigma(x_0,r)$, where $m(\sigma(x_0,r))$ denotes Lebesgue measure of the ball $\sigma(x_0,r)$.

The function u is called L^q-differentiable of order m at x_0 if there exists the polynomial $P(x)$ of degree at most m such that

$$\lim_{r\downarrow 0} r^{-m} V_q(u-P,x_0,r)=0$$

(see [11], [12], [6]).

We investigate L^q-mean limits at x_0 for the difference $R_\alpha f(x)-P(x)$ and find a condition for fulfilment of the limit equality

$$\lim_{r\downarrow 0} \upsilon(r)\, V_q(R_\alpha f-P,x_0,r)=0,$$

where υ is a weight function. The theorem on L^q-differentiability of anisotropic Riesz potentials is also given. All results generalize the corresponding results for classical Riesz potentialas obtained in [10].

Limit of mean value and differentiability. It is clear that the set of points in which the derivatives of the functions $R_\alpha(y)$ are not defined, belongs to the set $\Gamma=\{y\in R^n :.\ y=(y_1,...,y_{i-1},0,y_{i+1},...,y_n),\, i=\overline{1,n}\ \}$. The Lebesgue measure of the set Γ is zero, therefore the derivatives of any order $R_\alpha(y)$ are defined almost everywhere. Let $\mathcal{R}_i$ be a domain of definition of the function

$$\frac{\partial^l}{\partial y^l}R_\alpha(y)=\frac{\partial^{|l|}}{\partial y_1^{l_1},\partial y_2^{l_2}...\partial y_n^{l_n}}R_\alpha(y),$$

where $l=(l_1,l_2,...,l_n)$ is multiindex and $|l|=l_1+l_2+...+l_n$..

Lemma 1 ([3, lemma 2]). *Let $l=(l_1,l_2,...,l_n)$ be multiindex and $\frac{1}{\lambda_i}\in N$ or $\lambda_i>|l|$ for any $i=\overline{1,n}$.*

Then for any $y\in\mathcal{R}_i$

$$\left|\frac{\partial^l}{\partial y^l}R_\alpha(y)\right|\le M\,\|y\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum_{j=1}^{n} l_j\lambda_j},$$

where M is a positive constant, independent of y.

Assume

$$Q_{\alpha,m,x_0}(x,y)=R_\alpha(x-y)-\sum_{|l|\le m}\frac{(x-x_0)^l}{l!}\frac{\partial^l}{\partial x^l}R_\alpha(x-y).$$

Let f be a measurable nonnegative function in R^n. Let's accept the notation

$$R_{\alpha,m,x_0}f(x)=\int_{R^n}Q_{\alpha,m,x_0}(x,y)\,f(y)\,dy.$$

Then assuming $b=2^{\frac{|\lambda|}{\lambda n}+1}$ and $c=2^{-\frac{|\lambda|}{\lambda n}-1}$ we can write

$$R_{\alpha,m,x_0}f(x)=R_1(x)+R_2(x)+R_3(x),$$

where

$$R_1(x) = \int\limits_{R^n \backslash \sigma\left(x_0, b\|x-x_0\|_\lambda\right)} Q_{\alpha,m,x_0}(x,y)\, f(y)\, dy,$$

$$R_2(x) = \int\limits_{\sigma\left(x_0, c\|x-x_0\|_\lambda\right)} Q_{\alpha,m,x_0}(x,y)\, f(y)\, dy,$$

$$R_3(x) = \int\limits_{\sigma\left(x_0, b\|x-x_0\|_\lambda\right) \backslash \sigma\left(x_0, c\|x-x_0\|_\lambda\right)} Q_{\alpha,m,x_0}(x,y)\, f(y)\, dy,$$

For each fixed x we denote by $\mathcal{R}(x)$ a domain of definition of $Q_{\alpha,m,x_0}(x,\cdot)$.

Lemma 2. *Let*

$$m \text{ be a nonnegative integer number and } \frac{1}{\lambda_i} \in N \text{ or } \frac{1}{\lambda_i} > m. \tag{3}$$

Then for any $y \in \sigma(x_0, c\|x-x_0\|_\lambda) \cap \mathcal{R}(x)$

$$Q_{\alpha,m,x_0}(x,y) < M \|x-x_0\|_\lambda^{\frac{\overline{\lambda} nm}{|\lambda|}} \|y-x_0\|_\lambda^{\frac{\overline{\lambda} nm}{|\lambda|}}.$$

Proof. Since $\|y-x_0\|_\lambda < c\|x-x_0\|_\lambda$,, then by lemma 1

$$\|x-y\|_\lambda \geq 2^{-\frac{|\lambda|}{\underline{\lambda} n}} \|x-x_0\|_\lambda - \|y-x_0\|_\lambda > c\|x-x_0\|_\lambda.$$

The lemma is proved.

Lemma 3. *Let (3) be satisfied. Then for any*

$$y \in \left(\sigma(x_0, b\|x-x_0\|_\lambda) \backslash \sigma(x_0, c\|x-x_0\|_\lambda)\right) \cap \mathcal{R}(x)$$

the following estimate is valid

$$|Q_{\alpha,m,x_0}(x,y)| \leq M \|y-x_0\|_\lambda^{\alpha-n}.$$

Proof. Since $c\|x-x_0\|_\lambda \leq \|y-x_0\|_\lambda < b\|x-x_0\|_\lambda$, then

$$\|x-y\|_\lambda \leq 2^{-\frac{|\lambda|}{\underline{\lambda} n}} \left(\|x-x_0\|_\lambda + \|y-x_0\|_\lambda\right) \leq 2^{-\frac{|\lambda|}{\underline{\lambda} n}} (b+1) \|y-x_0\|_\lambda.$$

By lemma 1 we have

$$|Q_{\alpha,m,x_0}(x,y)| \leq \|y-x_0\|_\lambda^{\alpha-n} + \sum_{|l|\leq m} \frac{\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum_{i=1}^n \lambda_i l_i}}{l!} \left(M_1 \|y-x_0\|_\lambda^{\alpha-n-\sum_{i=1}^n \lambda_i l_i} \right) \leq$$

$$\leq M \|y-x\|_\lambda^{\alpha-n}.$$

The lemma is proved.

Lemma 4 ([3] lemma 3). *Let (3) be satisfied. Then for any*

$$y \in \left(R^n \backslash \sigma(x_0, b\|x-x_0\|_\lambda)\right) \cap \mathcal{R}(x)$$

the estimate

$$\left|Q_{\alpha,m,x_0}(x,y)\right| \leq M \left\|x-x_0\right\|_\lambda^{\frac{\lambda n(m+1)}{|\lambda|}} \left\|y-x_0\right\|_\lambda^{\alpha-n-\frac{\lambda n(m+1)}{|\lambda|}}$$

is valid.

Further on, we'll always assume that v is a positive nonincreasing function in $(0,\infty)$ and there exists a positive constant $A>0$ that for any $r>0$

$$(v_1) \qquad v(r) \leq Av(2r).$$

Lemma 5. *Let (3) be satisfied, let f be a measurable nonnegative function in R^n, there exists $\delta_1 \geq 0$ such that*

$$(v_2) \qquad r^{\frac{\lambda n(m+1)}{|\lambda|}+\delta_1} v(r) \quad \text{is a nondecreasing function in } (0,\infty),$$

for which

$$\int_{R^n} \left\|y-x_0\right\|_\lambda^{\alpha-n+\delta_1} v\left(\left\|y-x_0\right\|_\lambda\right) f(y)\,dy < \infty. \tag{4}$$

Then

$$\left\|x-x_0\right\|_\lambda^{\delta} v\left(\left\|x-x_0\right\|_\lambda\right) R_1(x) = O(1), \quad as \quad \left\|x-x_0\right\|_\lambda \to \infty.$$

If additionally

$$(v_3) \qquad \lim_{r\downarrow\infty} r^{\frac{\lambda n(m+1)}{|\lambda|}+\delta_1} v(r) = 0,$$

then

$$\left\|x-x_0\right\|_\lambda^{\delta} v\left(\left\|x-x_0\right\|_\lambda\right) R_1(x) = o(1), \quad as \quad \left\|x-x_0\right\|_\lambda \to \infty.$$

Proof. Let $\varepsilon>0$ and $b\left\|x-x_0\right\|_\lambda < \varepsilon$. Then by lemma 4 and by condition (v_2) we have

$$\left|R_1(x)\right| \leq M_1 \left\|x-x_0\right\|_\lambda^{\frac{\lambda n(m+1)}{|\lambda|}} \int_{R^n\setminus\sigma\left(x_0,b\|x-x_0\|_\lambda\right)} \left\|y-x_0\right\|_\lambda^{\alpha-n-\frac{\lambda n(m+1)}{|\lambda|}} f(y)\,dy \leq$$

$$\leq M_2 \left[\left\|x-x_0\right\|_\lambda^{\frac{\lambda n(m+1)}{|\lambda|}} \left(\varepsilon^{\frac{\lambda n(m+1)}{|\lambda|}+\delta_1} v(\varepsilon)\right)^{-1} \times \right.$$

$$\times \int_{R^n\setminus\sigma(x_0,\varepsilon)} \left\|y-x_0\right\|_\lambda^{\alpha-n+\delta_1} v\left(\left\|y-x_0\right\|_\lambda\right) f(y)\,dy +$$

$$+ \left(\left\|x-x_0\right\|_\lambda^{\delta_1} v\left(\left\|x-x_0\right\|_\lambda\right)\right)^{-1} \times$$

$$\left. \times \int_{\sigma(x_0,\varepsilon)\setminus\sigma\left(x_0,b\|x-x_0\|_\lambda\right)} \left\|y-x_0\right\|_\lambda^{\alpha-n+\delta_1} v\left(\left\|y-x_0\right\|_\lambda\right) f(y)\,dy \right].$$

By (4) we obtain

$$|R_1(x)| \le \left(\|x-x_0\|_\lambda^{\delta_1} v\left(\|x-x_0\|_\lambda\right)\right)^{-1}\left[M_\varepsilon \|x-x_0\|_\lambda^{\frac{\lambda n(m+1)}{|\lambda|}+\delta_1} v\left(\|x-x_0\|_\lambda\right)+\right.$$

$$\left.+M_2 \int\limits_{\sigma(x_0,\varepsilon)} \|y-x_0\|_\lambda^{\alpha-n+\delta_1} v\left(\|y-x_0\|_\lambda\right) f(y)\,dy\right].$$

Hence it is obvious that

$$|R_1(x)| \le O\left(\left(\|x-x_0\|_\lambda^{\delta_1} v\left(\|y-x_0\|_\lambda\right)\right)^{-1}\right), \quad \text{as} \quad \|x-x_0\|_\lambda \to 0.$$

If (v_3) is satisfied, then

$$\limsup_{\|x-x_0\|_\lambda\to 0} \|x-x_0\|_\lambda^{\delta_1} v\left(\|x-x_0\|_\lambda\right)|R_1(x)| \le$$

$$\le M_2 \int\limits_{\sigma(x_0,\varepsilon)} \|y-x_0\|_\lambda^{\alpha-n+\delta_1} v\left(\|y-x_0\|_\lambda\right) f(y)\,dy.$$

By virtue of (4) and by arbitrariness of ε we obtain

$$\lim_{\|x-x_0\|_\lambda\to\infty} \sup \|x-x_0\|_\lambda^{\delta_1} v\left(\|x-x_0\|_\lambda\right)|R_1(x)| = 0.$$

The lemma is proved.

Lemma 6. *Let (3) be satisfied, f be a measurable nonnegative function in R^n and there exist $\delta_2 > 0$ such that*

$$(v_4) \qquad r^{\frac{\lambda n m}{|\lambda|}-\delta_2} v(r) \ \text{ is a nonincreasing function in } (0,\infty)$$

and

$$\int\limits_{\sigma(x_0,1)} \|y-x_0\|_\lambda^{\alpha-n-\delta_2} v\left(\|y-x_0\|_\lambda\right) f(y)\,dy < \infty. \tag{5}$$

Then

$$\|x-x_0\|_\lambda^{\delta_2} v\left(\|x-x_0\|_\lambda\right) R_2(x) = o(1), \quad \text{as } \|x-x_0\|_\lambda \to 0.$$

Proof. By lemma 2 and condition (v_4) we have

$$|R_2(x)| \le M \|x-x_0\|_\lambda^{\frac{\overline{\lambda} n m}{|\lambda|}} \int\limits_{\sigma\left(x_0, c\|x-x_0\|_\lambda\right)} \|y-x_0\|_\lambda^{\alpha-n-\frac{\overline{\lambda} n m}{|\lambda|}} f(y)\,dy \le$$

$$\le M \|x-x_0\|_\lambda^{\delta_2} v^{-1}\left(\|x-x_0\|_\lambda\right) \int\limits_{\sigma\left(x_0, c\|x-x_0\|_\lambda\right)} \|y-x_0\|_\lambda^{\alpha-n-\delta_2} v\left(\|y-x_0\|_\lambda\right) f(y)\,dy$$

and the proof of the lemma is over by (5).

Let the function w satisfy the following conditions

(w_1) w is a nonnegative nondecreasing function in $(0,\infty)$;

(w_2) there exists a positive constant $A_1>0$ that for any $r>0$

$$A_1^{-1}w(r)\le w\left(r^2\right)\le A_1 w(r).$$

If the conditions (w_1) and (w_2) are satisfied, then one can prove the fulfilment of the following conditions (see [8], [9], [10]).

(w_3) there exists the constant $A_2>0$ that for any $r>1$

$$A_2^{-1}w(r)\le w(2r)\le A_2 w(r);$$

(w_4) for any $\gamma>0$ there exists $A(\gamma)>1$ that for any $r>0$

$$A^{-1}(\gamma)\,w(r)\le w(r^\gamma)\le A(\gamma)\,w(r);$$

(w_5) if $\gamma>0$, then there exists $A>0$ that

$$s^\gamma w\left(s^{-1}\right)\le At^\gamma w\left(t^{-1}\right),\ \ 0<s<t.$$

Theorem 1. *Let (3) be satisfied,* $1<p\le\frac{n}{\alpha}$, $q\in(0,\infty)$ *and* $\frac{1}{q}\ge\frac{1}{p}-\frac{\alpha}{n}$. *Assume that* f *is a measurable nonnegative function in* R^n, v *is a positive nonincreasing function in* $(0,\infty)$, *the conditions* (v_2), (v_4), *(2), (4), (5) are satisfied,*

(v_5) *there exists* $\beta<\alpha$ *that* $\lim\limits_{r\downarrow 0} r^\beta v(r)=0$

and

$$\lim_{r\downarrow 0}\left[r^{n-\alpha p-p\max(\delta_1,\delta_2)}v^{-p}(r)\,w\left(r^{-1}\right)\right]^{-1}\int\limits_{\sigma(x_0,r)} f^p(y)\,w(f(y))\,dy=0, \qquad (6)$$

where w *is a function satisfying the conditions* (w_1) *and* (w_2). *Then*

$$r^{\max(\delta_1,\delta_2)}v(r)\,V_q\left(R_{\alpha,m,x_0}f(x),x_0,r\right)=O(1),\ \ as\ \ r\downarrow 0. \qquad (7)$$

If additionally $\beta\le\frac{\underline{\lambda}n(m+1)}{|\lambda|}+\delta_1$, *then*

$$r^{\max(\delta_1,\delta_2)}v(r)\,V_q\left(R_{\alpha,m,x_0}f(x),x_0,r\right)=o(1),\ \ as\ \ r\downarrow 0. \qquad (8)$$

Remark 1. It is obtained from the conditions (v_1) and (v_4) that $\frac{\overline{\lambda}nm}{|\lambda|}-\delta_2\le\frac{\underline{\lambda}n(m+1)}{|\lambda|}+\delta_1$; It is obvious from the conditions (v_4) and (v_5) that $\beta>\frac{\overline{\lambda}nm}{|\lambda|}-\delta_2$ and hence $\frac{\overline{\lambda}nm}{|\lambda|}-\delta_2<\alpha$. If $\frac{\underline{\lambda}n(m+1)}{|\lambda|}+\delta_1<a$ then (v_5) is obtained from (v_2) for β satisfying the condition $\frac{\underline{\lambda}n(m+1)}{|\lambda|}+\delta_1<\beta<a$.

Proof. If $\beta < \frac{\lambda n(m+1)}{|\lambda|}$ then (v_3) is obtained from (v_5). Then by lemma 5 and 6 it is sufficient to prove the theorem for $R_3(x)$.

We accept the notation $E(x) = E(x, x_0) = \sigma(x_0, b\|x - x_0\|_\lambda) \backslash \sigma(x_0, c\|x - x_0\|_\lambda)$. Then by lemma 3 for $\mu > 0$ we can write

$$R_3(x) \le M \int\limits_{E(x)} \|x - y\|_\lambda^{\alpha - n} f(y)\, dy =$$

$$= M \left[\int\limits_{\left\{y \in E(x): f(x) > \|x - x_0\|_\lambda^{-\mu}\right\}} \|x - y\|_\lambda^{\alpha - n} f(y)\, dy + \right.$$

$$\left. + \int\limits_{\left\{y \in E(x): f(x) \le \|x - x_0\|_\lambda^{-\mu}\right\}} \|x - y\|_\lambda^{\alpha - n} f(y)\, dy \right] = M\left[R_{3,1}(x) + R_{3,2}(x)\right].$$

By the condition (w_4) we obtain that for $f(y) > \|x - x_0\|_\lambda^{-\mu}$

$$w(f(y)) \ge w\left(\|x - x_0\|_\lambda^{-\mu}\right) \ge Aw\left(\|x - x_0\|_\lambda^{-1}\right).$$

Let γ be such that $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$. Then $\alpha - \gamma = n\left(\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}\right) \ge 0$.
If $\|x - x_0\|_\lambda < r < 1$,, then

$$R_{3,1}(x) \le M_1 w^{-\frac{1}{p}}\left(\|x - x_0\|_\lambda^{-1}\right) \int\limits_{E(x)} \|x - y\|_\lambda^{\alpha - n} f(y)\, w^{\frac{1}{p}}(f(y))\, dy \le$$

$$\le M_1 w^{-\frac{1}{p}}\left(\|x - x_0\|_\lambda^{-1}\right)\left(b\|x - x_0\|_\lambda\right)^{\alpha - \gamma} \int\limits_{E(x)} \|x - y\|_\lambda^{\gamma - n} f(y)\, w^{\frac{1}{p}}(f(y))\, dy \le$$

$$\le M_1 w^{-\frac{1}{p}}\left(r^{-1}\right) r^{\alpha - \gamma} \int\limits_{E(x)} \|x - y\|_\lambda^{\gamma - n} f(y)\, w^{\frac{1}{p}}(f(y))\, dy.$$

If $y \in E(x)$, then

$$\|x - y\|_\lambda \ge 2^{\frac{|\lambda|}{\lambda n}} \|x - x_0\|_\lambda - \|x_0 - y\|_\lambda \ge \left(2^{\frac{|\lambda|}{\lambda n}}\left(2^{\frac{|\lambda|}{\lambda n}} - 1\right)^{-1} - 1\right) \times$$

$$\times \|x_0 - y\|_\lambda = \|x_0 - y\|_\lambda.$$

We pass to the spherical coordinates. We use the transformation $x_i = \theta_i \rho^{\frac{|\lambda|}{\lambda n}}$, where θ_i are the coordinates of the point θ on a unit sphere $S_{n-1} = \{x : \|x\|_\lambda = 1\}$.

It is known that the Jacobian of this transformation equals $\rho^{n-1}\Omega(\theta)$, where $\Omega(\theta)$ depends only on the angle θ (see [1]). Let $M_3 = \int\limits_{S_{n-1}} \Omega(\theta)\, d\theta$. Then

$$R_{3,2}(x) \le \|x-x_0\|_\lambda^{-\mu} \int\limits_{\sigma\left(x_0, b\|x-x_0\|_\lambda\right)} \|x-y\|_\lambda^{\alpha-n}\, dy \le$$

$$\le \|x-x_0\|_\lambda^{-\mu} \int\limits_{\sigma\left(x_0, b\|x-x_0\|_\lambda\right)} \|x_0-y\|_\lambda^{\alpha-n}\, dy = M_3 \|x_0-y\|_\lambda^{\alpha-\mu}.$$

By the Minkowsky inequality

$$V_q\left(R_3(x), x_0, r\right) \le M_3 r^{\alpha-\mu} + M_2 \left(\frac{1}{m\sigma(x_0,r)}\right)^{\frac{1}{q}} w^{\frac{1}{q}}\left(r^{-1}\right) r^{\alpha-\gamma} \times$$

$$\times \left\{ \int\limits_{\sigma(x_0,r)} \left(\int\limits_{\sigma(x_0,br)} \|x-y\|_\lambda^{\gamma-n} f(y)\, w^{\frac{1}{p}}(f(y))\, dy \right)^q dx \right\}^{\frac{1}{q}}.$$

We apply the Hardy-Littlewood –Sobolev inequality (see [2]) to the last integral:

$$r^{\max(\delta_1,\delta_2)} v(r)\, V_q\left(R_3(x), x_0, r\right) \le M_3 r^{\alpha-\mu+\max(\delta_1,\delta_2)} v(r) +$$

$$+ M_4 \left[r^{n-\alpha p-\max(\delta_1,\delta_2)} v^{-p}(r)\, w\left(r^{-1}\right)\right]^{-\frac{1}{p}} \left(\int\limits_{\sigma(x_0,br)} f^p(y)\, w(f(y))\, dy \right)^{\frac{1}{p}}.$$

Now we choose $\mu > 0$ such that $\beta < \alpha - \mu$. Then by (6) and (v_5)

$$\lim_{r\downarrow 0} r^{\max(\delta_1,\delta_2)} v(r)\, V_q\left(R_3(x), x_0, r\right) = 0.$$

The theorem is proved.

For $p > 1$ we assume

$$w^*(r) = \left(\int\limits_0^r w\left(t^{-1}\right)^{-\frac{1}{p-1}} t^{-1} dt \right)^{\frac{p-1}{p}}.$$

If $\alpha p = n$, then in theorem 1 $q = \infty$. For this case the following theorem is valid.

Theorem 2. *Let* $p = \dfrac{n}{a} > 1$, f *be a measurable nonnegative function in* R^n, v *be a positive nonincreasing function in* $(0,\infty)$, *conditions (2), (3), (4), (5), (6),* (v_2), (v_4) *be satisfied, and condition* (v_5) *be satisfied at some* $\beta < \alpha - \max(\delta_1,\delta_2)$. *If* $w^*(1) < \infty$, *then*

$$\sup_{x\in\sigma(x_0,r)} \left|R_{\alpha,m,x_0} f(x)\right| = o\left(v^{-1}(r)\, r^{-\max(\delta_1,\delta_2)} w^{\frac{1}{p}}\left(r^{-1}\right) w^*(r)\right), \quad as \quad r \downarrow 0. \quad (9)$$

Remark 2. It is clear that

$$w^*(r) \geq \left(\int_{r^2}^{r} w^{-\frac{1}{p-1}} \left(t^{-1}\right) t^{-1} dt \right)^{\frac{p-1}{p}} \geq M w^{-\frac{1}{p}} \left(r^{-1}\right) \ln^{\frac{p-1}{p}} \left(r^{-1}\right).$$

Hence

$$\lim_{r \downarrow \infty} w^*(r) \, w^{\frac{1}{p}} \left(r^{-1}\right) = \infty.$$

So from (9) it doesn't imply

$$r^{\max(\delta_1, \delta_2)} v(r) \sup_{x \in \sigma(x_0, r)} \left| R_{\alpha, m, x_0} f(x) \right| = O(1), \quad \text{as} \ \ r \downarrow \infty.$$

Proof. By lemmas 5 and 6 it is sufficient to prove the theorem for $R_3(x)$. We choose δ satisfying the condition $0 < \delta < \alpha - \max(\delta_1, \delta_2) - \beta$.

Then by lemma 3 we can write

$$R_3(x) \leq M_1 \int_{E(x)} \|x - y\|_\lambda^{\alpha - n} f(y) \, dy =$$

$$= M \left[\int_{\left\{y \in E(x) : f(y) > \|y - x\|_\lambda^{-\delta}\right\}} \|x - y\|_\lambda^{\alpha - n} f(y) \, dy + \right.$$

$$\left. + \int_{\left\{y \in E(x) : f(y) \leq \|y - x\|_\lambda^{-\delta}\right\}} \|x - y\|_\lambda^{\alpha - n} f(y) \, dy \right] = M_1 [I + II],$$

where $E(x) = E(x, x_0) = \sigma(x_0, b \|x - x_0\|_\lambda) \backslash \sigma(x_0, c \|x - x_0\|_\lambda)$.

By Hōlder inequality we have

$$I \leq \left(\int_{\left\{y \in E(x) : f(y) > \|y - x\|_\lambda^{-\delta}\right\}} f^p(y) \, w(f(y)) \, dy \right)^{\frac{1}{p}} \times$$

$$\times \left(\int_{\left\{y \in E(x) : f(y) \leq \|y - x\|_\lambda^{-\delta}\right\}} \|x - y\|_\lambda^{\frac{(\alpha - n)p}{p-1}} w^{-\frac{1}{p-1}} (f(y)) \, dy \right)^{\frac{p-1}{p}}.$$

By condition (w_4) we obtain that for $f(y) > \|y - x\|_\lambda^{-\delta}$

$$w(f(y)) \geq w\left(\|y - x\|_\lambda^{-\delta} \right) \geq M_2 w\left(\|y - x\|_\lambda^{-1} \right).$$

Let $\|x-x_0\|_\lambda < r$. If $y \in E(x)$, then

$$\|x-y\|_\lambda \le 2^{\frac{|\lambda|}{\lambda n}} \left(\|x-x_0\|_\lambda + \|x_0-y\|_\lambda\right) \le 2^{\frac{|\lambda|}{\lambda n}} (1+b) \|x-x_0\|_\lambda = er,$$

where $e = 2^{\frac{|\lambda|}{\lambda n}} (1+b)$.

Then

$$I \le M_3 \left(\int\limits_{\sigma\left(x_0, b\|x-x_0\|_\lambda\right)} f^p(y)\, w(f(y))\, dy \right)^{\frac{1}{p}} \times$$

$$\times \left(\int\limits_{\sigma(x,er)} \|x-y\|_\lambda^{\frac{(\alpha-n)p}{p-1}} w^{-\frac{1}{p-1}} \left(\|y-x\|_\lambda^{-1}\right) dy \right)^{\frac{p-1}{p}}.$$

We pass to spherical coordinates in the second integral. Since $\frac{(\alpha-n)p}{p-1} + n - 1 = -1$, then

$$I \le M_4 \left(\int\limits_{\sigma\left(x_0, b\|x-x_0\|_\lambda\right)} f^p(y)\, w(f(y))\, dy \right)^{\frac{1}{p}} \left(\int\limits_0^{er} \rho^{-1} w^{-\frac{1}{p-1}} \left(\rho^{-1}\right) d\rho \right)^{\frac{p-1}{p}} \le$$

$$\le M_5 \left(\int\limits_{\sigma\left(x_0, b\|x-x_0\|_\lambda\right)} f^p(y)\, w(f(y))\, dy \right)^{\frac{1}{p}} w^*(r).$$

Passing also to spherical coordinates

$$II \le M_6 \|x-x_0\|_\lambda^{\alpha-\delta} \le M_6 r^{\alpha-\delta}.$$

As a result

$$R_3(x) \le M \left(w^*(r) \left(\int\limits_{\sigma\left(x_0, b\|x-x_0\|_\lambda\right)} f^p(y)\, w(f(y))\, dy \right)^{\frac{1}{p}} + r^{\alpha-\delta} \right) =$$

$$= M r^{\max(\delta_1,\delta_2)} v^{-1}(r)\, w^*(r)\, w^{\frac{1}{p}}\left(r^{-1}\right) \times$$

$$\times \left(r^{-\max(\delta_1,\delta_2)} v(r)\, w^{-\frac{1}{p}}\left(r^{-1}\right) \left(\int\limits_{\sigma(x_0,br)} f^p(y)\, w(f(y))\, dy \right)^{\frac{1}{p}} + r^{\alpha-\delta-\max(\delta_1,\delta_2)} v(r) \right).$$

Since $\alpha - \delta - \max(\delta_1, \delta_2) < \beta$, then we obtain the proof of the theorem from conditions (v_5) and (6).

Let $k(r)$ be a nonincreasing positive function in $(0, \infty)$. For the set $E \in R^n$ and for the open set $G \in R$ we define the quantity

$$C_{k,\Phi_p}(E, G) = \inf \int_G \Phi_p(g(y)),$$

where $\Phi_p(t) = t^p w(t)$, $t > 0$, w is a function satisfying conditions (w_1), (w_2) and the lower bound is taken on all measurable and nonnegative functions g in G for which

$$\int_{R^n} k(\|x - y\|_\lambda) g(y)\, dy \geq 1, \quad x \in E$$

(see [5], [8], [10]).

If for any bounded open set G

$$C_{k,\Phi_p}(E \cap G, G) = 0,$$

then for simplification we'll write $C_{k,\Phi_p}(E) = 0$.

If $k(\|x\|_\lambda) = \|x\|_\lambda^{\beta - n}$, then instead of C_{k,Φ_p} we'll write C_{β,Φ_p}. If some property is satisfied out of the set E and $C_{k,\Phi_p}(E) = 0$, then it is called that this property is satisfied C_{k,Φ_p}-quasi everywhere.

Let $p > 1$. Denote by Φ_w^p a class of measurable and nonnegative functions f in R^n for which

$$\int_{R^n} \Phi_p(f(y))\, dy < \infty,$$

where $\Phi_p(t) = t^p w(t)$, for $t > 0$ and w is a function satisfying conditions (w_1), (w_2).

Lemma 7. *Let $k(r)$ be a nonincreasing positive function in $(0, \infty)$, $f \in \Phi_w^p$. Then*

$$C_{k,\Phi_p}(E_f) = 0,$$

where $E_f = \left\{ x : \int_{R^n} k(\|x - y\|_\lambda) f(y)\, dy = \infty \right\}$.

Proof. For any $a > 1$ we assume

$$E_{f,a} = \left\{ x : \int_{R^n} k(\|x - y\|_\lambda) f(y)\, dy \geq a \right\}.$$

Then for any $x \in E_{f,a}$

$$\int_{R^n} k(\|x - y\|_\lambda) \frac{f(y)}{a} dy \geq 1.$$

By definition of C_{k,Φ_p}

$$C_{k,\Phi_p}(E_{f,a} \cap G, G) \leq \int_G \Phi_p\left(\frac{f(y)}{a}\right) dy \leq a^{-p} \int_G f^p(y)\, w(f(y))\, dy.$$

Since $E_f \subset E_{f,a}$, then it follows from arbitrariness of a that

$$C_{k,\Phi_p}(E_f) = 0.$$

The lemma is proved.

Lemma 8 *([10, lemma 4.4]). Let f be a measurable nonnegative function in R^n, v be a positive nonincreasing function in $(0,\infty)$, conditions $(v_2),(v_4)$, (v_5) be satisfied. Then*

$$C_{k,\Phi_p}(E) = 0.$$

where $k(r) = r^{\alpha-n}v(r)$, $\quad 0<\alpha<n$,

$$E = \left\{ x_0 : \limsup_{r\downarrow 0} \left[r^{n-\alpha p} v^{-p}(r)\, w\left(r^{-1}\right)\right]^{-1} \int\limits_{\sigma(x_0,r)} \Phi_p(f(y))\,dy > 0 \right\}.$$

Theorem 3. *Let (3) be satisfied, $1<p\le \frac{n}{\alpha}$, $q\in(0,\infty)$ and $\frac{1}{q} \ge \frac{1}{p} - \frac{\alpha}{n}$. Assume that $f\in \Phi_w^p$, v is a positive nonincreasing function in $(0,\infty)$, conditions (2), (v_2), (v_4), (v_5) are satisfied, where $\delta_1+\delta_2 \le \frac{\overline{\lambda} nm}{|\lambda|}$. Then (7) is satisfied for C_{k,Φ_p}- quasi every $x_0\in R^n$, where $k(r) = r^{\alpha-n-\delta_2}v(r)$. If additionally $\beta \le \frac{\underline{\lambda} n(m+1)}{|\lambda|} + \delta_1$, then (8) is satisfied for C_{k,Φ_p}- quasi every $x_0\in R^n$.*

Proof. Assume

$$E_1 = \left\{ x : \int\limits_{R^n} \|y-x\|_\lambda^{\alpha-n+\delta_1} v(\|y-x\|_\lambda) f(y)\,dy = \infty \right\};$$

$$E_2 = \left\{ x : \int\limits_{\sigma(x,1)} \|y-x\|_\lambda^{\alpha-n-\delta_2} v(\|y-x\|_\lambda) f(y)\,dy = \infty \right\};$$

$$E_3 = \left\{ x : \limsup_{r\downarrow 0} \left[r^{n-\alpha p - p\max(\delta_1,\delta_2)} v^{-p}(r)\, w\left(r^{-1}\right)\right]^{-1} \int\limits_{\sigma(x_0,r)} \Phi_p(f(y))\,dy > 0 \right\}.$$

Let $E = E_1\cup E_2\cup E_3$. It is clear that for $x_0\in R^n\backslash E$ the conditions of the theorem are satisfied. So, it is sufficient to prove that $C_{k,\Phi_p}(E)=0$.

By lemma 7 $C_{k,\Phi_p}(E_2)=0$.

By conditions (v_4) and $\delta_1+\delta_2\le \frac{\overline{\lambda} nm}{|\lambda|}$ we have

$$\int\limits_{\|y-x\|_\lambda\ge 1} \|y-x\|_\lambda^{\alpha-n+\delta_1} v\left(\|y-x\|_\lambda\right) f(y)\,dy \le$$

$$\le \int\limits_{\|y-x\|_\lambda\ge 1} \|y-x\|_\lambda^{\alpha-n+\delta_1+\delta_2-\frac{\overline{\lambda} nm}{|\lambda|}} f(y)\,dy \le$$

$$\le v(1) \int\limits_{\|y-x\|_\lambda\ge 1} \|y-x\|_\lambda^{\alpha-n} f(y)\,dy<\infty.$$

Then

$$E_1=\left\{x:\int\limits_{\|y-x\|_\lambda<1} \|y-x\|_\lambda^{\alpha-n+\delta_1} v\left(\|y-x\|_\lambda\right) f(y)\,dy=\infty\right\}\subset$$

$$\subset\left\{x:\int\limits_{\|y-x\|_\lambda<1} \|y-x\|_\lambda^{\alpha-n-\delta_2} v\left(\|y-x\|_\lambda\right) f(y)\,dy=\infty\right\}=E_2.$$

So $C_{k,\Phi_p}(E_1)=0$.

$$E_3\subset\left\{x:\ \limsup_{r\downarrow 0}\left[r^{n-(\alpha-\delta_2)p} v^{-p}(r)\, w\left(r^{-1}\right)\right]^{-1} \int\limits_{\sigma(x,r)} \Phi_p\left(f(y)\right) dy>0\right\}.$$

By lemma 8 $C_{k,\Phi_p}(E_3)=0$. By subadditivity of C_{k,Φ_p} (see [9, lemma 5.1]) $C_{k,\Phi_p}(E)=0$.

The theorem is proved.

Theorem 4. *Let (3) be satisfied,* $1<p\le\frac{n}{\alpha}$, $q\in(0,\infty)$ *and* $\frac{1}{q}\ge\frac{1}{p}-\frac{\alpha}{n}$. *Assume that* $f\in\Phi_w^p$, v *is a positive nonincreasing function in* $(0,\infty)$, *conditions (2),* (v_2), (v_4), (v_5) *are satisfied, as well as* $\frac{\underline{\lambda} n(m+1)}{|\lambda|}<\alpha$ *and there exists* $\delta_3>0$ *and* $t>0$ *that for any* $r\in(0,t)$

$$r^{\alpha-\delta_2-\delta_3} v(r)<M,$$

where M *is a positive constant number. Then for almost every* $x_0\in R^n$ *(7) is satisfied. If additionally* $\beta\le\frac{\underline{\lambda} n(m+1)}{|\lambda|}$, *then for almost every* $x_0\in R^n$ *(8) is satisfied.*

Proof. By condition (v_2)

$$\int\limits_{\|y-x\|_\lambda<1} \|y-x\|_\lambda^{\alpha-n+\delta_1} v\left(\|y-x\|_\lambda\right) f(y)\,dy \le$$

$$\leq v(1)\int\limits_{\|y-x\|_\lambda<1}\|y-x_0\|_\lambda^{\alpha-n-\frac{\lambda n(m+1)}{|\lambda|}}f(y)\,dy.$$

Since $\frac{\lambda n(m+1)}{|\lambda|}<\alpha$, then

$$\int\limits_{\|y-x_0\|_\lambda\geq1}\|y-x_0\|_\lambda^{\alpha-n_1-\frac{\lambda n(m+1)}{|\lambda|}}f(y)\,dy\leq\int\limits_{\|y-x_0\|_\lambda\geq1}\|y-x_0\|_\lambda^{\alpha-n}f(y)\,dy<\infty.$$

Then for almost every $x_0\in R^n$

$$\int\limits_{R^n}\|y-x_0\|_\lambda^{\alpha-n-\frac{\lambda n(m+1)}{|\lambda|}}f(y)\,dy<\infty.$$

So, for almost every $x_0\in R^n$ (4) is satisfied

$$\int\limits_{\sigma(x_0,1)}\|y-x\|_\lambda^{\alpha-n-\delta_2}v(\|y-x_0\|_\lambda)f(y)\,dy\leq M\int\limits_{\sigma(x_0,1)}\|y-x\|_\lambda^{\delta_3-n}f(y)\,dy.$$

Hence, we obtain that for almost all $x_0\in R^n$ (5) is satisfied

$$\lim_{r\downarrow0}\left[r^{n-\alpha p-p\max(\delta_1,\delta_2)}v^{-p}(r)\,w\left(r^{-1}\right)\right]^{-1}\int\limits_{\sigma(x_0,r)}\Phi_p(f(y))\,dy=$$

$$=\lim_{r\downarrow0}r^{\alpha+\max(\delta_1,\delta_2)p}v^p(r)\lim_{r\downarrow0}\frac{1}{r^n}\int\limits_{\sigma(x_0,r)}\Phi_p(f(y))\,dy.$$

By condition (v_5) and Lebesgue theorem with respect to Lebesgue points for almost every $x_0\in R^n$ (6) is satisfied. So, for almost everywhere $x_0\in R^n$ the conditions of theorem 1 are satisfied. This completes the proof.

Theorem 5. *Let (3) be satisfied* $1<p\leq\frac{n}{\alpha}$, $q\in(0,\infty)$ *and* $\frac{1}{q}\geq\frac{1}{p}-\frac{\alpha}{n}$. *Assume that* $f\in\Phi_w^p$, *condition (3) is satisfied and there exist the numbers* δ_1 *and* δ_2 *that* $\delta_1\geq0$, $\delta_2\geq0$ *and*

$$\delta_1\leq\frac{\lambda nm}{|\lambda|}-\delta_2<\min\left(\frac{\lambda n(m+1)}{|\lambda|}+\delta_1,\alpha\right).$$

Then for $C_{\alpha-\delta_2-\gamma,\Phi_p}$*- quasi every* $x_0\in R^n$

$$r^{\max(\delta_1,\delta_2)-\gamma}V_q\left(R_{\alpha,m,x_0}f(x),x_0,r\right)=o(1),\quad\text{as }\ r\downarrow\infty,$$

where $\frac{\overline{\lambda}nm}{|\lambda|}-\delta_2<\gamma<\min\left(\frac{\lambda n(m+1)}{|\lambda|}+\delta_1,\alpha\right)$.

Proof. Taking $v(r)=r^{-\gamma}$ and $\gamma<\beta<\min\left(\frac{\lambda n(m+1)}{|\lambda|}+\delta_1,\alpha\right)$ we can see that the conditions of theorem 3 are satisfied.

Theorem 6. *Let the conditions of theorem 3 be satisfied and additionally* $m < \min\left(\frac{\underline{\lambda} n\,(m+1)}{|\lambda|} + \delta_1, \alpha\right) - \max(\delta_1, \delta_2)$*. Then potential (1) is L^q-differentiable in m power C_{η,Φ_p} -quasi every, where* $\eta = \alpha - \delta_2 - \max\left(m, \frac{\overline{\lambda} nm}{|\lambda|} - \delta_2\right)$.

Theorem 7. *Let (3) be satisfied,* $p = \frac{n}{\alpha} > 1$*. Assume that* $f \in \Phi^p_w$, $w^*(1) < \infty$ *is a positive nonincreasing function in* $(0,\infty)$*, conditions (2),* (v_2), (v_4), (v_5) *are fulfilled, where* $\delta_1 + \delta_2 \le \frac{\overline{\lambda} nm}{|\lambda|}$*, and* $\beta < \alpha - \max(\delta_1, \delta_2)$*. Then for* C_{k,Φ_p}*- quasi every* $x_0 \in R^n$ *(9) is satisfied, where* $k(r) = r^{\alpha - n - \delta_2} v(r)$.

Proof. By the method of the proof of theorem 3 it is proved that the conditions of theorem 2 are satisfied for C_{k,Φ_p}*- quasi every* $x_0 \in R^n$.

Theorem 8. *Let (3) be satisfied,* $p = \frac{n}{\alpha} > 1$*. Assume that* $f \in \Phi^p_w$, $w^*(1) < \infty$, v *is a positive nonincreasing function in* $(0,\infty)$*, conditions 2),* (v_2), (v_4), (v_5), $\max\left(\frac{\underline{\lambda} n\,(m+1)}{|\lambda|}, \beta + \max(\delta_1, \delta_2)\right) < \alpha$ *are satisfied and additionally there exists* $\delta_3 > 0$ *and* $t > 0$ *that for any* $r \in (0,t)$

$$r^{\alpha - \delta_2 - \delta_3} v(r) < M,$$

where M is a positive constant number. Then for almost every $x_0 \in R^n$ *(9) is satisfied.*

Theorem 9. *Let* $p = \frac{n}{\alpha} > 1$, $f \in \Phi^p_w$, $w^*(1) < \infty$ *(2) and (3) be satisfied. Assume that there exist the numbers* δ_1 *and* δ_2 *such that* $\delta_1 \ge 0$, $\delta_2 \ge 0$ *and*

$$\delta_1 \le \frac{\overline{\lambda} nm}{|\lambda|} - \delta_2 < \min\left(\frac{\underline{\lambda} n\,(m+1)}{|\lambda|} + \delta_1, \alpha - \max(\delta_1, \delta_2)\right).$$

Then for $C_{\alpha - \delta_2 - \gamma, \Phi_p}$*- quasi every* $x_0 \in R^n$

$$\sup_{x \in \sigma(x_0, r)} |R_{\alpha, m, x_0} f(x)| = o\left(r^{\gamma - \max(\delta_1,\delta_2)} w^{\frac{1}{p}}\left(r^{-1}\right) w^*(r)\right), \quad \text{as} \quad r \downarrow 0\,,$$

where $\frac{\overline{\lambda} nm}{|\lambda|} - \delta_2 < \gamma < \min\left(\frac{\underline{\lambda} n\,(m+1)}{|\lambda|} + \delta_1, \alpha - \max(\delta_1, \delta_2)\right)$.

Proof. Taking $v(r) = r^{-\gamma}$ we can see that the conditions of theorem 7 are satisfied.

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