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NOTE ON ALGEBRAS OF POWER SERIES IN ELEMENTS OF A LIE ALGEBRA

Abstract

In this note we prove the algebra of power series in elements of a finite-dimensional Lie algebra in a montel space.

1. Introduction.

In this note we prove that the Fréchet, Arens-Michael algebra $\mathrm{O}_{\mathfrak{g}}(D)$ of holomorphic functions (or convergent power series) in noncommuting variables generating a finite-dimensional Lie algebra $\mathfrak{g}$ on absolutely convex domain D from the dual space $\mathfrak{g}^*$ is a montel space. Let us remind that a locally convex space X is said to be a *montel space* if each bounded subset in X precompact. For the algebras $\mathcal{O}(D)$ of usual holomorphic functions in several complex variables $z_1, ..., z_n$ on arbitrary domains $D \subseteq \mathbf{C}^n$ that result is well known (see for instance [3, item 1.1.13]) and belongs to so-called mathematical folklore. A noncommutative version of algebra $\mathcal{O}(D)$ in terms of Lie algebra generators was suggested in [1], [2]. By replacing the complex variables $z_1, ..., z_n$ with elements of a certain finite-dimensional Lie algebra $\mathfrak{g}$, the algebra $\mathcal{O}_{\mathfrak{g}}(D)$ is defined [2] as a certain Frechet completion of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In contrast to function alge-bras, the algebras $\mathcal{O}_{\mathfrak{g}}(D)$ are not neither commutative nor semisimple. For instance, if $\mathfrak{g}$ is a solvable Lie algebra then $\mathcal{O}_{\mathfrak{g}}(D)$ has the nontrivial Jacobson radical Rad $\mathcal{O}_{\mathfrak{g}}(D)$, which is a closed (left or right) ideal generated by the commutant $[\mathfrak{g}, \mathfrak{g}]$ and the quotient algebra $\mathcal{O}_{\mathfrak{g}}(D) / \mathrm{Rad}\mathcal{O}_{\mathfrak{g}}(D)$ is reduced to $\mathcal{O}\ (D \cap \Delta(\mathfrak{g}))$, where $\Delta(\mathfrak{g}) = \{\lambda \in \mathfrak{g}^* : \lambda([\mathfrak{g}, \mathfrak{g}]) = \{0\}\}$ is the space of all Lie characters on $\mathfrak{g}$. Moreover, $\mathcal{O}_{\mathfrak{g}}(D) = \mathcal{O}(D)$ whenever $\mathfrak{g}$ is a commutative Lie algebra. We focus in the following question. Does the property to by montel space remain true for noncommutative algebras $\mathcal{O}_{\mathfrak{g}}(D)$ too? The positive answer to this question was announced in [2], but did not prove out of prescribed aims of that paper. In this note we fill this gap and suggest more detailed proof.

2. Strongly increasing norm sets

First, we briefiy remind the construction of algebras $\mathcal{O}_{\mathfrak{g}}(D)$. Let L be a complex finite-dimensionallinear space, $M \subseteq L$ and let $F \subseteq L^*$. The *F-convex hull* $\widehat{M_F}$ of M is defined as a set of those $z \in L$ such that

$$|\lambda(z)| \leq \sup\{|\lambda(w)| : w \in M\}$$

for all $\lambda \in F$. A domain (that is, open connected subset) D in L is called *F-convex*, if $\widehat{K}_F \subset D$ for each compact subset $K \subset D$. On the ground of the well known identification $L^{**} = L$, one can define M-convex hull $\widehat{F}_M$ of a subset $F \subseteq L^*$. A domain $D \subseteq L$ is said to be absolutely convex if D is stable under taking finite absolutely convex combinations. Let K be a compact subset L with non-empty interior int$K$. Then $\widehat{K}_{L^*} = abc\,(K)$ [2, Lemma 4.1], where $abc\,(K)$ is the absolutely convex hull of K. In particular, a domain $D \subseteq L$ is L^*-convex iff D is an absolutely convex.

Now let $\mathfrak{p} = \{p_i\}_{i\in\Lambda}$ be a saturated norm set on L. We say that $\mathfrak{p}$ is *strongly increasing* if for each $i \in \Lambda$ there exists $k \in \Lambda$ such that $p_i\,(x) < p_k\,(x)\,,\ x \neq 0$ (in this case we write $p_i << p_k$). One can define an absolutely convex domain in L^* by each strongly increasing norm set on L . Namely, let $\mathfrak{p}$ be a strongly increasing norm set on L and let M_i be the unit ball L^* with respect to the dual norm p_i^*. Take $\lambda \in M_i$ and assume that $p_i << p_k$. Then $p_i \leq \epsilon_{ik} p_k$, where

$$\sup\{|\lambda\,(x)| : p_k\,(x) \leq 1\} \leq \epsilon_{ik} < 1,$$

whence $\lambda \in int\ M_k$. Thereby, $M_i \subseteq int\ M_k$. We set $D = \cup_{i\in\Lambda} M_i$. It is clear that D is a domain in L^* called a *domain associated by* $\mathfrak{p}$. Let K be a compact set in D. It is beyond a doubt $D = \cup_{i\in\Lambda} int\ M_i$. Moreover, $K \subseteq M_i$ for a certain $i \in \Lambda$ for $\mathfrak{p}$ is a saturated norm set. Thereby, $abc\,(K) \subseteq M_i$, that is, D is an absolutely convex domain.

Conversely, let D be an arbitrary absolutely convex domain in L^*. There exists a strongly increasing norm set $\mathfrak{p}$ on L such that its associated domain coincides with D [2, Lemma 4.2].

Now let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathfrak{p} = \{p_i\}_{i\in\Lambda}$ a strongly increasing norm set on $\mathfrak{g}$ associated by a domain $D \subseteq \mathfrak{g}^*$ and let $\mathcal{A}\,(p_i) = \mathcal{A}\,(\mathfrak{g})$ be the Banach enveloping algebra of the Banach-Lie algebra $\mathfrak{g}$ furnished with the norm p_i [2, item 2.1]. The family of Banach algebra $\{\mathcal{A}\,(p_i)\}$ with their connecting bounded algebra homomorphisms u_{ik}: $\mathcal{A}\,(p_k) \to \mathcal{A}\,(p_i)\,,\ p_i << p_k$ (u_{ik} induces the identity map on the enveloping algebra $\mathcal{U}\,(\mathfrak{g})$) generates a projective system $\mathcal{A}\,(\mathfrak{g}_\mathfrak{p})$ $= \{\{\mathcal{A}\,(p_i)\}\,, \{u_{ik}\}\,, \Lambda\}$ of Banach algebras. Let $\underleftarrow{\mathcal{A}\,(\mathfrak{g}_\mathfrak{p})}$ be its inverse limit. By definition, we set $\mathcal{O}_\mathfrak{g}\,(D) = \underleftarrow{\mathcal{A}\,(\mathfrak{g}_\mathfrak{p})}$. The soundness of the latter definition is confirmed in [2].

3. The main result

Now we prove the main result of this note.

Theorem 3.1. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra and let* D *be an absoluiely convex domain in* $\mathfrak{g}^*$. *The algebra* $\mathcal{O}_\mathfrak{g}\,(D)$ *is a Montel space.*

Proof. With $\mathcal{O}_\mathfrak{g}\,(D) = \varprojlim \{\{\mathcal{A}\,(p_i)\}\,, \{u_{ik}\}\,, \Lambda\}$ in mind, one suffices to demonstrate that for each $i \in \Lambda$ there corresponds $k \in \Lambda,\ p_i << p_k$, such that the connecting morphism $u_{ik} : \mathcal{A}\,(p_k) \to \mathcal{A}\,(p_i)$ is a compact operator. By definition (see [2]), $\mathcal{A}\,(p_k) = l_1\,(\mathfrak{g}_k)\,/J_k$, where

$$l_1\,(\mathfrak{g}_k) = \left\{ (y_n) \in \prod_{n\in\mathbf{Z}_+} \mathfrak{g}^{\otimes n} : \sum_{n\in\mathbf{Z}_+} p_k^{(n)}\,(y_n) < \infty \right\}$$

is the Banach algebra of all absolutely convergent sequences, $\mathfrak{g}^{\otimes n} = \mathfrak{g} \otimes ... \otimes \mathfrak{g}$ (n times), $p_k^{(n)}$ is the projective n-times tensor product of the norm p_k on itself, J_k is a closed two-sided ideal in $l_1(\mathfrak{g}_k)$ generated by tensors

$$x \otimes y - y \otimes x - [x, y], \quad x, y \in \mathfrak{g} \ .$$

Therefore, one suffices to prove the compactness of the embedding $l_1(\mathfrak{g}_k) \to l(\mathfrak{g}_i)$. Let $S = \{x : p_k(x) = 1\}$. Then

$$q = \sup\{p_i(x) : x \in S\} < 1$$

and $p_i \leq q \cdot p_k$. Take a bounded sequence $\left\{f_k = \sum_{m \in \mathbf{Z}_+} x_{km}\right\}_{k \in \mathbf{N}}$ in $l_1(\mathfrak{g}_k)$. let us denote the norm on $l_1(\mathfrak{g}_k)$ by p_k too, and let $C = \sup_k p_k(f_k)$. then

$$p_i(x) \leq q^m p_k(x), \quad x \in \mathfrak{g}^{\otimes m}.$$

For each m, we have a bounded sequence $\{x_{km}\}_{k \in \mathbf{N}}$ in $\mathfrak{g}^{\otimes m}$. Each of them has a convergent subsequence. Using diagonal process one can construct a subsequence $\{f_j\}_{j \in \mathbf{N}}$ coordinate-wisely convergent to a certain sequence $f = (x_0, x_1, ..., x_n, ...)$. It remains to prove that $\{f_j\}_{j \in \mathbf{N}}$ is a Cauchy sequence in $l_1(\mathfrak{g}_i)$. For $j,\ j', m$ we deduce that

$$p_k\left(x_{jm} - x_{j'm}\right) \leq p_k(x_{jm}) + p_k\left(x_{j'm}\right) \leq p_k(f_j) + p_k\left(f_{j'}\right) \leq 2C.$$

Take $\varepsilon > 0$ and $i \in \mathbf{N}$ so that $\sum_{m \geq i} q^m < \varepsilon/4C$. For fixed i one can find $M \geq 1$ such that $\sum_{m<i} p_i\left(x_{jm} - x_{j'm}\right) < \varepsilon/2$ whenever $j, j' \geq M$. Then

$$p_i\left(f_j - f_{j'}\right) = \sum_{m<i} p_i\left(x_{jm} - x_{j'm}\right) + \sum_{m \geq i} p_i\left(x_{jm} - x_{j'm}\right) <$$

$$< \varepsilon/2 + \sum_{m \geq i} q^m p_k\left(x_{jm} - x_{j'm}\right) < \varepsilon$$

for $j, j' \geq M$, that is, $\{f_j\}_{j \in \mathbf{N}}$ is a Cauchy sequence, and we end the proof.

References

[1]. Dosiev A.A. *Holomorphic functions of a basis of a nilpotent Lie algebra.* Funct. Anal., and its Appl., 34 (4), 2000, pp.302-304.

[2]. Dosiev A.A. *Algebras of power series in elements of a Lie algebra and Slodkowski spectra.* Zapiski nauch. sem. POMI, 290, 2002, pp.72-121. (Russian).

[3]. Range R.M. *Holomorphic functions and Integral representations in several complex variables.* Springer-Verlag, 1990.

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