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THE A-INTEGRAL AND CAUCHY-GREEN'S FORMULA

Abstract

Let G be a simply connected bounded domain on the complex plane C, let $\gamma = \partial G$, and assume that γ is a closed rectifiable Jordan curve. Denote by m the Lebesgue linear measure on γ.

We consider the following problems:

1) find conditions on $F(z)$, defined on G and having a finite nontangential boundary value $F(t)$ for m-almost all $t \in \gamma$, the fucntion $F(t)$ is A-integrable and just the formula, analogue of formula Cauchy-Green.

2) find conditions on $F(z)$, analytical on G, $F(z)$ representable on G as a Cauchy A-integral.

1. Let G be a bounded simply connected domain of the complex plane $\mathbf{C}$, whose boundary is a closed rectifiable Jordan curve γ. Denote by m the Lebesgue linear measure on γ. The complex valued fucntion f on γ, measurable (with regard to measure m) is called A-integrable on γ, if

$$m\{t \in \gamma : |f(t)| > \lambda\} = o\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty$$

and there exists the finite limit

$$(A)\int_{\gamma} f(t)\,dt \overset{def}{=} \lim_{\lambda \to +\infty} \int_{\{t\in\gamma:|f(t)|\le\lambda\}} f(t)\,dt\ .$$

For the fucntion F, given in G, and for $\alpha \in (1;+\infty)$ we'll define the fucntion $F^*_\alpha(t)$ on γ, having supposed

$$F^*_\alpha(t) = \sup\{|F(z)| : \ z \in G, \ \ |z - t| < \alpha\rho(z,\gamma)\},$$

if the set $\{z \in G : \ |z - t| < \alpha\rho(z,\gamma)\}$ is not empty, and $F^*_\alpha(t) = 0$, otherwise, where $\rho(z,\gamma)$ is an Euclidean distance from z to γ. The fucntion F^*_α is a natural analogue of nontangential maximum fucntion (see [1, chapter 1, §4, p.30) in case of the functions defined in domains of the plane $\mathbf{C}$.

In the paper the following problems are considered: under which conditions on the fucntion $F(z)$, defined in the domain G and having the finite angular boundary values $F(t)$ almost everywhere (with regard to measure m) on γ, the fucntion $F(t)$ is A-integrable on γ, and the formula, similar to the Cauchy-Green's integral formula is true.

If G is a circle, and F is representable in G by the Cauchy type integral, then A-integrability on γ of the boundary values of the fucntion F follows from Titchmarsh's result [2] on A-integrability of adjoint fucntion, and representability of F

in G by Cauchy A-integral has been proved by P.L.Ulyanov [3]. Then P.L.Ulyanov [4], [5] established the truth of these results in case of domains with smooth boundary, satisfying some additional condition. Later these problems for functions from V.I.Smirnov's class N^+ were considered by A.B.Alexandrov [6]. In the paper [7] T.S.Salimov found the condition on the analytical in G fucntion $F(z)$, at which $F(z)$ has the finite angular boundary values $F(t)$ almost everywhere (with regard to measure m) on γ, the fucntion $F(t)$ is A-integrable on γ, and $F(z)$ is representable in G by Cauchy A-integral.

Let us formulate the basic results of the present paper:

Theorem 1. *Let the function F be defined on the domain G and satisfy the following conditions:*

1) F have continuous partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in G, where

$$\iint_G \left|\frac{\partial F}{\partial \bar{\xi}}\right| dxdy < +\infty, \qquad \xi = x + iy;$$

2) for some $\alpha \in (1; +\infty)$

$$m\{t \in \gamma : F_\alpha^*(t) > \lambda\} = o\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty;$$

3) for $m-$ almost all $t \in \gamma$ there exists the finite angular boundary value $F(t)$.
Then

a)

$$(A) \int_\gamma F(t)\,dt = 2i \iint_G \frac{\partial F}{\partial \bar{\xi}} dxdy; \tag{1}$$

b) for all $z \in G$

$$F(z) = \frac{1}{2\pi i}(A)\int_\gamma \frac{F(t)}{t-z}dt - \frac{1}{\pi}\iint_G \frac{\partial F}{\partial \bar{\xi}}\,\frac{dxdy}{\xi - z}\,. \tag{2}$$

Corollary. *Let the function F be analytical in G and at some $\alpha \in (1; +\infty)$*

$$m\{t \in \gamma : F_\alpha^*(t) > \lambda\} = o\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty.$$

Then F has the finite angular boundary value $F(t)$ for $m-$almost all $t \in \gamma$ and

a) $(A) \int_\gamma F(t)\,dt = 0;$

b) $F(z) = \frac{1}{2\pi i}(A)\int_\gamma \frac{F(t)}{t-z}dt, \quad z \in G.$

Remark. In the case $\alpha > 2$ and under additional conditions in the case $\alpha \in (1; 2]$ the corollary was proved by T.S.Salimov [7].

2. Construct the subset $G(E,\alpha)$ of the domain G, which will be used in the proof of theorem 1, and we'll study some its properties, where $\alpha > 1$, E is an open subset of γ, such that $E \neq 0$ and $E \neq \gamma$.

Denote $P = \gamma \backslash E$

$$r = \frac{\alpha - 1}{3\alpha}, \quad \alpha_0 = \frac{2\alpha + 1}{\alpha + 2}, \; \delta_0 = \frac{1}{2} \arccos \frac{1}{\alpha_0}, \; n_0 = \left[\frac{\pi}{\delta_0} \right] + 1, \tag{3}$$

where $[x]$ is an entire part of the number $x \in R$.

Then, using Bezikovich theorem on overlappings [8, chapter 1, §1, p.13], from the system of circles $\{B(t, r\rho(t,P))\}_{t\in E}$ we'll choose at most countable number of the circles $\{B_q\}_{q\in Q}$ $(B_q = B(t_q, r\rho(t_q,P)), \;\; q \in Q)$ such that

$$E \subset \bigcup_{q\in Q} B_q \tag{4}$$

and each point from the $\mathbf{C}$ is overapped by at most θ_0 circles from $\{B_q\}$, such that the number θ_0 is absolute constant. From the last it follows, that

$$\sum_{q\in Q} m(E \cap B_q) \leq \theta_0 mE \; .$$

Taking into account also, that $m(E \cap B_q) \geq 2r\rho(t_q, P), \;\; q \in Q$ hence we'll obtain

$$\sum_{q\in Q} \rho(t_q, P) \leq \frac{\theta_0}{2r} mE \; . \tag{5}$$

Let $l_k, \;\; k = \overline{0, n_0 - 1}$ be straight lines in the plane $\mathbf{C}$, given by the equations

$$x \, \sin\left(\frac{\pi}{n_0} k\right) - y \, \cos\left(\frac{\pi}{n_0} k\right) = 0, \quad k = \overline{0, n_0 - 1}$$

respectively.

For every point $t_q, \;\; q \in Q$ we'll divide the plane $\mathbf{C}$ into $2n_0$ sectors with the help of straight lines, parallel to $l_k, \;\; k = \overline{0, n_0 - 1}$, and passing through the point t_q. Denote these sectors by $S_q^{(k)}, \;\; k = \overline{1, 2n_0}$ respectively

Let

$$P_q^{(k)} = P \cap \overline{S_q^{(k)}}, \quad k = \overline{1, 2n_0} \; .$$

Since, the set $P_q^{(k)}$ is closed, it is possible to take such a point $t_q^{(k)} \in P_q^{(k)}$, $k = \overline{1, 2n_0}$ that

$$\rho\left(t_q, P_q^{(k)}\right) = \left| t_q - t_q^{(k)} \right|, \quad k = \overline{1, 2n_0} \; .$$

Denoting

$$\tau_q^{(k)} = t_q + \frac{t_q - t_q^{(k)}}{\alpha_0^2 - 1}, \quad k = \overline{1, 2n_0},$$

we'll assume

$$K_q = \bigcap_{k=1}^{2n_0} B\left(\tau_q^{(k)}, \frac{\alpha_0}{\alpha_0^2 - 1} \left| t_q - t_q^{(k)} \right| \right), \quad q \in Q$$

and

$$G\left(E,\alpha\right)=G\backslash\bigcup_{q\in Q}K_q. \tag{6}$$

Lemma 1. *If* $z\in G\left(E,\alpha\right)$*, then* $\rho\left(z,P\right)<\alpha\rho\left(z,\gamma\right)$*.*

Proof. Let's choose the point $t\in\gamma$ such that $\rho\left(z,\gamma\right)=\left|z-t\right|$. In the case $t\in P$, we'll obtain, that $\rho\left(z,P\right)=\left|z-t\right|=\rho\left(z,\gamma\right)<\alpha\rho\left(z,\gamma\right)$. Now consider the case $t\in E$. By virtue of (4) $t\in B_q$ at some $q\in Q$, hence for any $k=\overline{1,2n_0}$

$$\left|t_q-t\right|\leq r\rho\left(t_q,P\right)\leq r\rho\left(t_q,P_q^{(k)}\right)=r\left|t_q-t_q^{(k)}\right|\ .$$

Then, by virtue of $z\notin K_q$ (see, (6)) there exists such $k_0\in\{1,2,...,2n_0\}$ that

$$z\notin B\left(\tau_q^{(k_0)},\frac{\alpha_0}{\alpha_0^2-1}\left|t_q-t_q^{(k_0)}\right|\right). \tag{7}$$

With the help of elementary calculations it is possible to show, that the locus ξ satisfying the inequality.

$$\left|\xi-t_q^{(k_0)}\right|\geq\alpha_0\left|\xi-t_q\right|,$$

is the circle $B\left(\tau_q^{(k_0)},\frac{\alpha_0}{\alpha_0^2-1}\left|t_q-t_q^{(k_0)}\right|\right)$. Thus from (7) it follows, that

$$\left|z-t_q^{(k_0)}\right|<\alpha_0\left|z-t_q\right|. \tag{8}$$

By virtue of triangle inequality

$$\left|z-t_q\right|<\left|z-t\right|+\left|t_q-t\right|\leq\left|z-t\right|+r\left|t_q-t_q^{(k_0)}\right|\leq$$

$$\leq\left|z-t\right|+r\left\{\left|z-t_q\right|+\left|z-t_q^{(k_0)}\right|\right\}<\left|z-t\right|+r\left(1+\alpha_0\right)\left|z-t_q\right|\ .$$

Hence

$$\left|z-t_q\right|<\frac{1}{1-r\left(1+\alpha_0\right)}\left|z-t\right|\ . \tag{9}$$

Taking into account, that $\left|z-t\right|=\rho\left(z,\gamma\right)$ and $\rho\left(z,P\right)\leq\left|z-t_q^{(k_0)}\right|$, by virtue of (3), (8) and (9) we have

$$\rho\left(z,P\right)\leq\left|z-t_q^{(k_0)}\right|<\frac{\alpha_0}{1-r\left(1+\alpha_0\right)}\rho\left(z,\gamma\right)=\alpha\rho\left(z,\gamma\right),$$

Q.E.D. (quod erat demomstrandum)

Lemma2. *Let* $t\in P$ *and at the point* t *there exist a tangent to* γ*. Then there exists such a number* $\varepsilon>0$ *that for all points* z*, lying on interior normal to* G *at the point* t *and satisfying the inequality* $\left|z-t\right|<\varepsilon$*, then inclusion* $z\in G\left(E,\alpha\right)$ *holds.*

Proof. Choose $\varepsilon_1>0$ such that if the point z lies on the interior normal to G at the point t and $\left|z-t\right|<\varepsilon_1$, then $z\in G$. Choose also $\varepsilon_2>0$ such that if $\tau\in\gamma$ and $\left|\tau-t\right|<\varepsilon_2$, then the angle between tangent at the point t and the ray $(t\tau)$ is less than δ_0.

Assume, that $\varepsilon = \min\left\{\varepsilon_1, \dfrac{\alpha_0-1}{\alpha_0}\varepsilon_2\right\}$. Let the point z lie on the interior normal to G at the point t and $|z-t|<\varepsilon$. Then $|z-t|<\varepsilon_1$ and thus $z\in G$. For proof of the lemma, by virtue of (6) it is sufficient to check, that $z\notin K_q$ for all $q\in Q$. Since for any $q\in Q$ the plane $\mathbf{C}$ is divided into $2n_0$ sectors $S_q^{(k)}$, then there exists such a number $k_1\in\overline{1,2n_0}$, that $t\in S_q^{(k_1)}$. Denote the angle between tangent at the point t and chord $[t;t_q]$ by φ_1, and the angle between straight lines (tt_q) and $\left(t_q t_q^{(k_1)}\right)$ by φ_2. Two cases are possible.

1) The case $|t-t_q|<\varepsilon_2$ (see fig.1). Then by virtue of $t\in S_q^{(k_1)}$ it follows, that $\varphi_1<\delta_0$ and $\varphi_2<\delta_0$ respectively. Let l be a tangent at the point t. Since $\delta_0<\dfrac{\pi}{6}$ at any $\alpha>1$, then we have

$$\left|z-\tau_q^{(k_1)}\right| \geq np_l\left[z;\tau_q^{(k_1)}\right] = np_l\left[t;t_q\right] + np_l\left[t_q;\tau_q^{(k_1)}\right] = |t_q-t|\cos\varphi_1 +$$

$$+\left|\tau_q^{(k_1)}-t_q\right|\cos\left(\varphi_1+\varphi_2\right) > \cos\left(2\delta_0\right)\left[|t_q-t| + \left|\tau_q^{(k_1)}-t_q\right|\right] >$$

$$> \frac{1}{\alpha_0}\left[\left|t_q-t_q^{(k_1)}\right| + \frac{1}{\alpha_0^2-1}\left|t_q-t_q^{(k_1)}\right|\right] = \frac{\alpha_0}{\alpha_0^2-1}\left|t_q-t_q^{(k_1)}\right|,$$

where $np_l\left[z_1;z_2\right]$ is the length of projection of the segment $[z_1;z_2]$ on the straight line l.

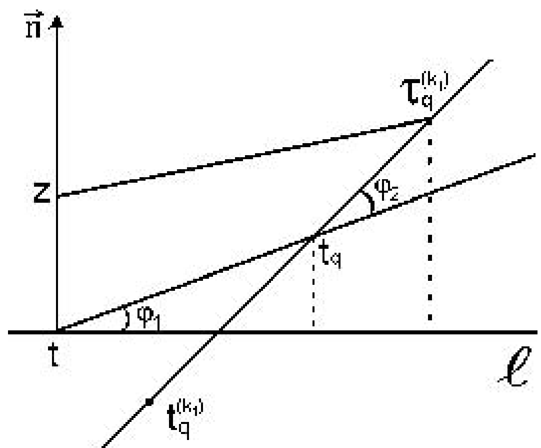

Fig.1

Hence, it follows, that $z\notin B\left(\tau_q^{(k_1)}, \dfrac{\alpha_0}{\alpha_0^2-1}\left|t_q-t_q^{(k_1)}\right|\right)$ and consequently, $z\notin K_q$

2) The case $|t-t_q|\geq\varepsilon_2$ (see fig.2).

We have

$$\left|z-\tau_q^{(k_1)}\right| \geq \left|t-\tau_q^{(k_1)}\right| - |t-z| \geq \left|t-\tau_q^{(k_1)}\right| - \frac{\alpha_0-1}{\alpha_0}\varepsilon_2 \geq$$

$$\geq \left|t-\tau_q^{(k_1)}\right| - \frac{\alpha_0-1}{\alpha_0}\left|t-t_q\right| \ . \qquad (10)$$

Let's denote by τ' the projection of the point $\tau_q^{(k_1)}$ on the straight line (tt_q).

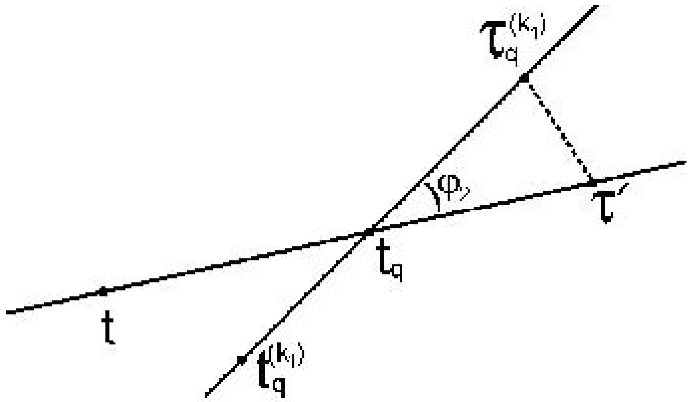

Fig.2

Since $\varphi_2 < \delta_0 < \dfrac{\pi}{6}$, then

$$\left|t-\tau_q^{(k_1)}\right| \geq \left|t-\tau'\right| = \left|t-t_q\right| + \left|t_q-\tau'\right| = \left|t-t_q\right| + \left|\tau_q^{(k_1)}-t_q\right|\cos\varphi_2 >$$

$$> \left|t-t_q\right| + \left|\tau_q^{(k_1)}-t_q\right|\cos\left(2\delta_0\right) = \left|t-t_q\right| + \frac{1}{\alpha_0}\left|\tau_q^{(k_1)}-t_q\right|.$$

Hence and from (10) it follows, that

$$\left|z-\tau_q^{(k_1)}\right| \geq \frac{1}{\alpha_0}\left[\left|t-t_q\right| + \left|\tau_q^{(k_1)}-t_q\right|\right] \geq \frac{\alpha_0}{\alpha_0^2-1}\left|t_q^{(k_1)}-t_q\right|,$$

and consequently, $z \notin K_q$.

Lemma 2 is proved.

Lemma 3. *$G\left(E,\alpha\right)$ is an open subset of the plane $\mathbf{C}$.*

Proof. Let $z \in G\left(E,\alpha\right)$. By virtue of (6) $z \in G$, and therefore $\rho\left(z,\gamma\right) > 0$. First of all we'll prove, that

$$K_q \subset B\left(t_q, \frac{1}{\alpha_0-1}\rho\left(t_q,P\right)\right) . \tag{11}$$

Really, for any point $z \in K_q$ by virtue of definition of the set K_q we have $z \in B\left(\tau_q^{(k)}, \frac{\alpha_0}{\alpha_0^2-1}\left|t_q-t_q^{(k)}\right|\right)$ at every $k=\overline{1,2n_0}$.

Then $\left|z-\tau_q^{(k)}\right| < \frac{\alpha_0}{\alpha_0^2-1}\left|t_q-t_q^{(k)}\right|$, and consequently,

$$\left|z-t_q\right| = \left|z-\tau_q^{(k)} + \frac{t_q-t_q^{(k)}}{\alpha_0^2-1}\right| < \left|z-\tau_q^{(k)}\right| + \frac{1}{\alpha_0^2-1}\left|t_q-t_q^{(k)}\right| <$$

$$< \frac{\alpha_0}{\alpha_0^2-1}\left|t_q-t_q^{(k)}\right| + \frac{1}{\alpha_0^2-1}\left|t_q-t_q^{(k)}\right| = \frac{1}{\alpha_0-1}\left|t_q-t_q^{(k)}\right| \leq \frac{1}{\alpha_0-1}\rho\left(t_q,P\right) .$$

Hence inclusion (11) follows.

As $\forall q \in Q \;\; t_q \in \gamma$, then from inequality (5) and inclusion (11) we obtain, that the circle $B\left(z,\rho\left(z,\gamma\right)/2\right)$ can be intersected only with finite number of sets from

$\{K_q\}_{q\in Q}$. As the sets K_q, $q\in Q$ are closed, then it follows, that at some $\delta>0$ the circle $B(z,\delta)$ is intersected with non of the sets K_q, $q\in Q$. Then, by virtue of $z\in G$ and (6) it follows, that some neighbourhood of the point z is contained in $G(E,\alpha)$.

Lemma 3 is proved.

Lemma 4. *The following correlations*

$$\partial G(E,\alpha)\cap E=\varnothing \quad \text{and} \quad \partial G(E,\alpha)\subset P\cup\left(\underset{q\in Q}{\cup}\partial K_q\right)$$

hold.

Proof. Form definition (6) of the set $G(E,\alpha)$ and from lemma 3 it follows, that

$$\partial G(E,\alpha)\subset\gamma\cup\left(\underset{q\in Q}{\cup}\partial K_q\right).$$

Therefore, it is sufficient to prove, that $\partial G(E,\alpha)\cap E=\varnothing$. If $t\in E$, then by virtue of (4) $t\in B_q$ at some $q\in Q$. Then $|t-t_q|\le r\rho(t_q,P)$. As for any $k\in\{1,2,...,2n_0\}$ the inequality $\left|t_q-t_q^{(k)}\right|\ge\rho(t_q,P)$ is fulfilled, then we have

$$\left|t-\tau_q^{(k)}\right|\le|t-t_q|+\left|t_q-\tau_q^{(k)}\right|\le r\rho(t_q,P)+$$

$$+\frac{1}{\alpha_0^2-1}\left|t_q-t_q^{(k)}\right|\le\left[r+\frac{1}{\alpha_0^2-1}\right]\left|t_q-t_q^{(k)}\right|<\frac{\alpha_0}{\alpha_0^2-1}\left|t_q-t_q^{(k)}\right|,$$

because by virtue of (3) $r<\frac{1}{\alpha_0+1}$. It means, that t is an interior point of the set K_q, and thus $t\notin\partial G(E,\alpha)$.

Lemma 4 is proved.

Let $\{G_i\}_{i\in I}$ be connectedness components of the set $G(E,\alpha)$. From construction of $G(E,\alpha)$ it follows, that for any $i\in I$ ∂G_i is a Jordan curve.

By virtue of lemma 4

$$\partial G_i\backslash\gamma\subset\underset{q\in Q}{\cup}\partial K_q\ , \tag{12}$$

$$\partial G_i\cap\gamma\subset P,\quad i\in I. \tag{13}$$

The set K_q is the intersection of $2n_0$ circles and therefore K_q is a convex set, and as one of these circles is of radius $\dfrac{\alpha_0}{\alpha_0^2-1}\rho(t_q,P)$ and K_q is a subset, of this circle, then the length of the boundary K_q is no more, than the length of the boundary of the considered circle, i.e. no more than $\dfrac{2\pi\alpha_0}{\alpha_0^2-1}\rho(t_q,P)$.

Therefore from (12) and (5) we obtain, that ∂G_i is rectifiable, $i\in I$.

If at the point $t\in P$ there exists a tangent to γ, then by virtue of lemma 2 $t\in\partial G_i$ at some $i\in I$, and, consequently

$$m\left(P\backslash\underset{i\in I}{\cup}\partial G_i\right)=0\ . \tag{14}$$

Lemma 5. *If $i, j \in I$ and $i \neq j$, then the intersection $\partial G_i \cap \partial G_j$ contains at most one point.*

Proof. From the construction of $G(E, \alpha)$ it follows, that $\partial G_i \cap \partial G_j$ doesn't contain the arc. Therefore, if this set contains at least two points, then the set $\mathbf{C} \backslash (G_i \cup G_j)$ will contain bounded connectedness component D. As $\partial D \subset \subset (\partial G_i \cup \partial G_j) \subset \overline{G}$, then from simply connectedness of G we obtain $D \subset G$. Besides $D \cap G(E, \alpha) \neq D$, since otherwise the set $G_i \cup G_j \cup D$ would be contained in some connectedness component of the set $G(E, \alpha)$, that is impossible. Then, from $D \subset G$, $D \cap G(E, \alpha) \neq D$ it follows, that $D \cap K_q \neq \varnothing$ at some $q \in Q$. Hence, by virtue of inclusion $D \subset G$ we obtain, that the intersection of ∂D with interior of K_q is not empty. This contradicts the correlations $\partial D \subset (\partial G_i \cup \partial G_j)$ and (6), that proves the lemma.

3. Now let's prove theorem 1.

Denote

$$E_\lambda = \{t \in \gamma : F^*_\alpha(t) > \lambda\}, \quad P_\lambda = \gamma \backslash E_\lambda.$$

If $E_\lambda = \varnothing$ at some $\lambda > 0$, then $|F(z)| \leq \lambda$ for all $z \in G$ and the truth of the theorem is established.

Let now $E_\lambda \neq \varnothing$ for all $\lambda > 0$. From definition of the function F^*_α it follows, that E_λ is an open subset of γ. Let $G(E_\lambda, \alpha)$ be a set, constructed in the previous item $\{G_i\}_{i \in I}$ be connectedness components of $G(E_\lambda, \alpha)$. Let's take on ∂G_i the positive orientation. (with regard to G_i). From Lemma 1 and definitions of the function F^*_α it follows, that $|F(z)| \leq \lambda$ as $z \in G(E_\lambda, \alpha)$. Therefore $|F(t)| \leq \lambda$ for almost all $t \in \partial G_i$ and it is true the equality

$$\int\limits_{\partial G_i} F(t)\, dt = 2i \iint\limits_{G_i} \frac{\partial F}{\partial \bar{\xi}} dxdy, \quad \xi = x + iy\ .$$

Hence

$$\left| \int\limits_{\partial G_i \cap \gamma} F(t)\, dt - 2i \iint\limits_{G_i} \frac{\partial F}{\partial \bar{\xi}} dxdy \right| = \left| \int\limits_{\partial G_i \backslash \gamma} F(t)\, dt \right| \leq \lambda \int\limits_{\partial G_i \backslash \gamma} |dt|\ .$$

Adding these inequalities, taking into account (12) and lemma 5 we have

$$\sum_{i \in I} \left| \int\limits_{\partial G_i \cap \gamma} F(t)\, dt - 2i \iint\limits_{G_i} \frac{\partial F}{\partial \bar{\xi}} dxdy \right| \leq \lambda \sum_{q \in Q} \int\limits_{\partial K_q} |dt|\ .$$

As the length o the boundary of the set K_q is no more than $\dfrac{2\pi\alpha_0}{\alpha_0^2 - 1}\rho(t_q, P)$, hence, applying (5) and (3) we obtain

$$\sum_{i \in I} \left| \int\limits_{\partial G_i \cap \gamma} F(t)\, dt - 2i \iint\limits_{G_i} \frac{\partial F}{\partial \bar{\xi}} dxdy \right| \leq \theta_0 \frac{3\pi\alpha_0\alpha}{(\alpha - 1)\left(\alpha_0^2 - 1\right)} \lambda m E_\lambda\ .$$

Later, by virtue of (13), (14) and lemma 5

$$\int_{P_\lambda} F(t)\,dt = \sum_{i\in I} \int_{\partial G_i\cap\gamma} F(t)\,dt\ .$$

From two last two correlations we get

$$\left|\int_{P} F(t)\,dt - 2i \iint_{G(E_\lambda,\alpha)} \frac{\partial F}{\partial\bar\xi} dxdy\right| \le \theta_0 \frac{3\pi\alpha_0\alpha}{(\alpha-1)\left(\alpha_0^2-1\right)}\lambda m E_\lambda\ . \tag{15}$$

Denote by P'_λ set of such points $t\in\gamma$, in which there exists the angular boundary value $F(t)$ and $|F(t)|\le\lambda$. From the conditions 3 of theorem 1 it follows, that $m\left(P_\lambda\backslash P'_\lambda\right)=0$.

Therefore

$$\int_{P'_\lambda} F(t)\,dt = \int_{P_\lambda} F(t)\,dt + \int_{P'_\lambda\backslash P_\lambda} F(t)\,dt\ .$$

On the other hand, we have

$$\left|\int_{P'_\lambda\backslash P_\lambda} F(t)\,dt\right| \le \lambda m\left(P'_\lambda\backslash P_\lambda\right) \le \lambda m\left(\gamma\backslash P_\lambda\right) = \lambda m E_\lambda\ . \tag{16}$$

Adding estimations (15) and (16) we obtain

$$\left|\int_{P'_\lambda} F(t)\,dt - 2i \iint_{G} \frac{\partial F}{\partial\bar\xi} dxdy\right| \le$$

$$\le \left[1+\theta_0 \frac{3\pi\alpha_0\alpha}{(\alpha-1)\left(\alpha_0^2-1\right)}\right]\lambda m E_\lambda + \iint_{G\backslash G(E_\lambda,\alpha)} \left|\frac{\partial F}{\partial\bar\xi}\right| dxdy\ .$$

As an area of the domain $G\backslash G(E_\lambda,\alpha)$ tends to zero as $\lambda\to+\infty$, then from here and from conditions 1 and 2 of theorem 1 it follows the item a) of theorem 1.

Let's prove item b). Let $\lambda>0$. Fix $z\in G$ and using the condition of the theorem we'll choose $\lambda_0>0$ such, that

$$mE_{\lambda_0} < \frac{1}{2}\min\left\{\left(\alpha_0^2-1\right)\rho(z,\gamma), m\gamma\right\}\ . \tag{17}$$

Then, let arbitrary $\lambda>\lambda_0$. The set E_λ is open, $E_\lambda\ne\varnothing$, and by virtue of (17) $E_\lambda\ne\gamma$. Applying the construction from the previous item (at $E=E_\lambda$) we obtain the system of the sets $\{K_q\}_{q\in Q}$ and the set $G(E_\lambda,\alpha)$. Let $\{G_i\}_{i\in I}$ be connectedness components $G(E_\lambda,\alpha)$. Orient ∂G_i positively (with regard of G_i), $i\in I$.

First of all we'll prove, that

$$\rho(z,K_q) > \frac{1}{2}\rho(z,\gamma),\ q\in Q. \tag{18}$$

Taking into account, that $K_q = \bigcap\limits_{k=1}^{2n_0} B\left(\tau_q^{(k)}, \frac{\alpha_0}{\alpha_0^2-1}\left|t_q - t_q^{(k)}\right|\right)$, then for any $k \in \{1,2,...,2n_0\}$ we have

$$\rho(z, K_q) = \rho\left(z, B\left(\tau_q^{(k)}, \frac{\alpha_0}{\alpha_0^2-1}\left|t_q - t_q^{(k)}\right|\right)\right) \geq$$

$$\geq \left|z - \tau_q^{(k)}\right| \geq |z - t_q| - \left|t_q - \tau_q^{(k)}\right| = |z - t_q| - \frac{1}{\alpha_0^2-1}\left|t_q - t_q^{(k)}\right| .$$

Let $\rho(t_q, P_\lambda) = \left|t_q - t_q^{(k_1)}\right|$. Since $|z - t_q| \geq \rho(z,\gamma)$ and $\rho(t_q, P_\lambda) \leq mE_\lambda$, then from here we obtain:

$$\rho(z, K_q) \geq \rho(z,\gamma) - \frac{1}{\alpha_0^2-1} mE_\lambda .$$

Remarking also, that by virtue of (17) $mE_\lambda < \frac{\alpha_0^2-1}{2}\rho(z,\gamma)$, we come to (18).

From (6) and (18) we have $z \in G(E_\lambda, \alpha)$. Besides from definition of the set P_λ and lemma 1 it follows, that $|F(z')| \leq \lambda$ as $z' \in G(E_\lambda, \alpha)$. Consequently, in $G(E_\lambda, \alpha)$ to the function F it is applicable the Cauchy-Riemann integral formula, that gives us

$$F(z) = \frac{1}{2\pi i}\sum_{i\in I}\int\limits_{\partial G_i}\frac{F(t)}{t-z}dt - \frac{1}{\pi}\iint\limits_{G(E_\lambda,\alpha)}\frac{\partial F}{\partial\bar{\xi}}\frac{dxdy}{\xi - z}\ , \quad \xi = x + iy .$$

Hence,

$$\left|F(z) - \frac{1}{2\pi i}\sum_{i\in I}\int\limits_{\partial G_i\cap\gamma}\frac{F(t)}{t-z}dt + \frac{1}{\pi}\iint\limits_{G}\frac{\partial F}{\partial\bar{\xi}}\frac{dxdy}{\xi - z}\right| \leq$$

$$\leq \frac{1}{2\pi}\sum_{i\in I}\int\limits_{\partial G_i\setminus\gamma}\frac{|F(t)|}{|t-z|}|dt| + \frac{1}{\pi}\iint\limits_{G\setminus G(E_\lambda,\alpha)}\left|\frac{\partial F}{\partial\bar{\xi}}\right|\frac{dxdy}{|\xi - z|} \leq$$

$$\leq \frac{\lambda}{2\pi}\sum_{i\in I}\int\limits_{\partial G_i\setminus\gamma}\frac{|dt|}{|t-z|} + \frac{1}{\pi}\iint\limits_{G\setminus G(E_\lambda,\alpha)}\left|\frac{\partial F}{\partial\bar{\xi}}\right|\frac{dxdy}{|\xi - z|} = J_1 + J_2 .$$

Estimate J_1 and J_2. Taking into account inclusion (12), Lemma (5) and inequality (18) for J_1 we have

$$J_1 \leq \frac{\lambda}{\pi\rho(z,\gamma)}\sum_{q\in Q}\int\limits_{\partial K_q}|dt| \leq \theta_0\frac{3\alpha_0\alpha}{(\alpha-1)\left(\alpha_0^2-1\right)\rho(z,\gamma)}\lambda mE_\lambda .$$

From here and from condition 2 of theorem 1 it implies, that J_1 tends to zero as $\lambda \to +\infty$.

Then, by virtue of inequality (18) for J_2 we obtain

$$J_2 \leq \frac{2}{\pi\rho(z,\gamma)}\iint\limits_{G\setminus G(E_\lambda,\alpha)}\left|\frac{\partial F}{\partial\bar{\xi}}\right|dxdy .$$

Since the area of the domain $G\backslash G(E_\lambda,\alpha)$ tends to zero as $\lambda\to+\infty$, then from condition 1 of theorem 1 it implies, that J_2 also tends to zero as $\lambda\to+\infty$.

Taking into account also, that by virtue of (13), (14) and lemma 5

$$\sum_{i\in I}\int\limits_{\partial G_i\cap\gamma}\frac{F(t)}{t-z}dt=\int\limits_{P_\lambda}\frac{F(t)}{t-z}dt\ ,$$

we obtain

$$F(z)=\lim_{\lambda\to+\infty}\frac{1}{2\pi i}\int\limits_{P_\lambda}\frac{F(t)}{t-z}dt-\frac{1}{\pi}\iint\limits_{G}\frac{\partial F}{\partial\bar{\xi}}\frac{dxdy}{\xi-z}\ . \tag{19}$$

As $m(P_\lambda\backslash P'_\lambda)=0$, then

$$\left|\int\limits_{P'_\lambda}\frac{F(t)}{t-z}dt-\int\limits_{P_\lambda}\frac{F(t)}{t-z}dt\right|=\left|\int\limits_{P'_\lambda\backslash P_\lambda}\frac{F(t)}{t-z}dt\right|\leq$$

$$\leq\frac{2\lambda}{\rho(z,\gamma)}m\left(P_\lambda\backslash P'_\lambda\right)\leq\frac{2\lambda}{\rho(z,\gamma)}mE_\lambda\ ,$$

and the last expression tends to zero as $\lambda\to+\infty$ by virtue of conditions 2 of theorem 1. Hence by virtue of (19) it follows equality (2).

The theorem is proved.

The corollary is directly obtained from theorem 1 and from the following lemma proved in the paper [7].

Lemma 6. *Let the function F be analytical in G and at some $\alpha\in(1;+\infty)$ $F^*_\alpha(t)$ be finite for all t from the measurable set $P\subset\gamma$. Then F has a finite angular boundary value $F(t)$ for almost all $t\in P$.*

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