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ON NORMAL SOLVABILITY OF BOUDNARY-VALUE PROBLEMS FOR ELLIPTIC TYPE FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATIONS

Abstract

In this paper the conditions providing the normal solvability of one class of fourth order operator-differential equations, when the boundary conditions contain the operators, are found.

Let H be a separable Hilbert space, A be a positive definite self-adjoint operator in H. It is known that the domain of operator A^{γ} $(\gamma > 0)$ becomes the Hilbert space H_{γ} with respect to scalar product $(x, y) = (A^{\gamma}x, A^{\gamma}y)\,,\ x, y \in D(A^{\gamma})$.

Denote by $L_2(R_+; H_{\gamma})$ a set of all the vector-functions with values from H_{γ} measurable by Bochner and for which $\|f\| = \left(\int\limits_0^{\infty} \|f(t)\|_{\gamma}^2\, dt\right)^{1/2} < \infty$. Further, let $L(X, Y)$ denote a set of linear bounded operators acting from the Hilbert space X to the other Y; $\sum(\cdot)$ be a spectrum of the operator $(\cdot)$; $\sum_{\infty}$ be an ideal of completely continuous operators in $L(H, H)$. Later on everywhere u', u'', u''' and $u^{(4)}$ are derivatives in terms of distribution theory [1].

Now we introduce the following sets

$$W_2^4(R_+; H) = \left\{u : u \in L_2(R_+; H_4)\,,\ u^{(4)} \in L_2(R_+; H)\right\},$$
$$W_2^{4,T,K}(R_+; H) =$$
$$= \left\{u : u \in W_2^4(R_+; H)\,,\ u(0) = Tu''(0)\,,\ u'(0) = Ku'''(0)\,,\right.$$
$$\left. T \in L\left(H_{3/2}; H_{7/2}\right),\ K \in L\left(H_{1/2}; H_{5/2}\right)\right\}.$$

Each of these sets provided by the norm

$$\|u\|_{W_2^4} = \left(\|u\|_{L_2(R_+;H_4)}^2 + \left\|u^{(4)}\right\|_{L_2(R_+;H)}^2\right)^{1/2}$$

becomes Hilbert space [1, p.29].

Now we pass to the statement of the boundary-value problem which we study.

Let $B_1, B_2, B_3 \in L(H; H)$, then the domain of operator-bundle

$$P(\lambda) = \lambda^4 E + \lambda^3 B_3 A + \lambda^2 B_2 A^2 + \lambda_1 B_1 A^3 + A^4 \qquad (1)$$

coincides with the space H_4; here E is a unit operator in H. Consider the operator-differential equation

$$P(d/dt)\,u = u^{(4)} + B_3 A u''' + B_2 A^2 u'' + B_1 A^3 u' + A^4 u = f, \quad t > 0 \qquad (2)$$

by fulfilling the boundary conditions

$$u(0) = Tu''(0), \ u'(0) = Ku'''(0), \ T \in L\left(H_{3/2}; H_{7/2}\right), \ K \in L\left(H_{1/2}; H_{5/2}\right), \quad (3)$$

where almost everywhere $f(t) \in H$, $u(t) \in H$. In this paper the sufficient conditions for normal solvability of boundary value problem (2),(3) are shown.

Definition 1. *Problem (2),(3) is called normal solvable if there exist the subspaces* $\tilde{L}_2(R_+; H) \subset L_2(R_+; H)$ *and* $\tilde{W}_2^{4,T,K}(R_+; H) \subset W_2^{4,T,K}(R_+; H)$ *with the finite-dimensional orthogonal complements in the spaces* $L_2(R_+; H)$ *and* $W_2^{4,T,K}(R_+; H)$*, respectively, and for any vector-funciton* $f(t) \in \tilde{L}_2(R_+; H)$ *there exists a unique vector-function* $u(t) \in \tilde{W}_2^{4,T,K}(R_+; H)$ *which satisfies equation (2) almost everywhere in* R_+*, boundary conditions (3) are satisfied in terms of convergence of norms of the spaces* $H_{7/2}, H_{5/2}$*, and the inequality*

$$\|u\|_{W_2^4} \le const \|f\|_{L_2} \quad (4)$$

holds.

At first we consider the following equation for the solution of the posed problem

$$P_0(d/dt)\, u \equiv u^{(4)} + A^4 u = f, \quad (5)$$

where $f(t) \in L_2(R_+; H)$. We denote by P_0 an operator acting from the space $W_2^{4,T,K}(R_+; H)$ to $L_2(R_+; H)$ by the following form: $P_0 u \equiv P_0(d/dt)\, u$, $u \in W_2^{4,T,K}(R+; H)$.

The following one is valid.

Theorem 1. *Let* $C = A^{7/2} T A^{-3/2}$, $S = A^{5/2} K A^{-1/2}$ *and these operators be commutative, i.e.* $CS = SC$*, and the point* $-1 \notin \sum (CS - S + C)$*. Then the operator* P_0 *realizes isomorphism from the space* $W_2^{4,T,K}(R_+; H)$ *to* $L_2(R_+; H)$*.*

Proof. It follows from the condition $-1 \notin \sum (CS - S + C)$ that the homogeneous equation $P_0(d/dt)\, u = 0$ has only zero solution from the space $W_2^{4,T,K}(R_+; H)$, and at any $f(t) \in L_2(R_+; H)$ the equation (5) has a solution from the space $W_2^{4,T,K}(R_+; H)$ represented in the form

$$\begin{aligned}
u(t) = {} & \frac{1}{4\sqrt{2}} \int_0^\infty \left((1+i)\, e^{-\frac{1+i}{\sqrt{2}}|t-s|A} + (1-i)\, e^{-\frac{1-i}{\sqrt{2}}|t-s|A} \right) A^{-3} f(s)\, ds - \\
& - \frac{i}{4\sqrt{2}} e^{-\frac{1+i}{\sqrt{2}} tA} A^{-7/2} (CS - S + C + E)^{-1} \times \\
& \times \left[\left((C + iE)(S - iE) + (E + iC)(S + iE) \right) \times \right. \\
& \left. \times A^{1/2} \int_0^\infty e^{-\frac{1+i}{\sqrt{2}} sA} f(s)\, ds + 2(E + iC)(S - iE)\, A^{1/2} \int_0^\infty e^{-\frac{1-i}{\sqrt{2}} sA} f(s)\, ds + \right] \\
& + \frac{i}{4\sqrt{2}} e^{-\frac{1-i}{\sqrt{2}} tA} A^{-7/2} (CS - S + C + E)^{-1} \times \\
& \times \left[2(E - iC)(S + iE)\, A^{1/2} \int_0^\infty e^{-\frac{1+i}{\sqrt{2}} sA} f(s)\, ds + \right. \\
& \left. + \left((E - iC)(S - iE) + (E + iC)(E - iS) \right) A^{1/2} \int_0^\infty e^{-\frac{1-i}{\sqrt{2}} sA} f(s)\, ds \right]
\end{aligned} \quad (6)$$

It is easily verified that the first member satisifies eqaution (5) and belongs to the space $W_2^4(R_+;H)$ (see [2,3]). Further, the inequalities [4, p.208]

$$\left\| A^{1/2} \int_0^\infty [\exp(-tA)] f(t)\,dt \right\|_H \le \frac{1}{\sqrt{2}} \|f\|_{L_2}, \tag{7}$$

$$\left\| A^{1/2} \int_0^\infty [\exp(-tA)] \psi \right\|_{L_2} \le \frac{1}{\sqrt{2}} \|\psi\|, \ \psi \in H, \tag{8}$$

imply the inequalities

$$\left\| A^{1/2} \int_0^\infty \left[\exp\left(-\frac{1 \pm i}{\sqrt{2}} tA \right) \right] f(t)\,dt \right\|_H \le \frac{1}{\sqrt[4]{2}} \|f\|_{L_2} \tag{9}$$

$$\left\| A^4 \left[\exp\left(-\frac{1 \pm i}{\sqrt{2}} tA \right) \right] \psi \right\|_{L_2} \le \frac{1}{\sqrt[4]{2}} \|\psi\|_{7/2}, \ \psi \in H_{7/2}. \tag{10}$$

Consequently, the second and third members in equality (6) also belong to the space $W_2^4(R_+;H)$. The fulfilment of boundary conditions (3) is verified immediately. The boundedness of the operator P_0 follows from the inequality

$$\|P_0 u\|_{L_2}^2 = \left\| u^{(4)} + A^4 u \right\|_{L_2}^2 \le 2 \|u\|_{W_2^4}^2. \tag{11}$$

Thus, the operator P_0 is bounded and one-to-one acts from the space $W_2^{4,T,K}(R_+;H)$ to $L_2(R_+;H)$, and by Banach inverse operator theorem realizes isomorphism by these spaces. The theorem is proved.

It follows from this theorem that $\|P_0 u\|_{L_2}$ is norm in the space $W_2^{4,T,K}(R_+;H)$ equivalent to original norm $\|u\|_{W_2^4}$. Now using the method of the paper [2], we prove the theorem on normal solvability of problem (2),(3).

Theorem 2. *Let* $-1 \notin \sum (CS - S + C)$, $CS = SC$, $A^{-1} \in \sum_\infty$, $B_i \in \sum_\infty$, $i = 1, 2, 3$, *and on imaginary axis the resolvent* $P^{-1}(\lambda)$ *exist. Then problem (2),(3) is normally solvable.*

Proof. Assuming $\upsilon = P_0 u$ in equation (2), and using representation (6) we obtain an integro-differential equation in the space $L_2(R_+;H)$ for definition of $\upsilon(t)$:

$$\upsilon(t) + \frac{1}{4\sqrt{2}} \left(\sum_{j=1}^3 B_j A^{4-j} \frac{d^j}{dt^j} \right) \int_0^\infty \left((1+i)\, e^{-\frac{1+i}{\sqrt{2}}|t-s|A} + (1-i)\, e^{-\frac{1-i}{\sqrt{2}}|t-s|A} \right) \times$$

$$\times A^{-3} \upsilon(s)\,ds - \frac{i}{4\sqrt{2}} \left(\sum_{j=1}^3 B_j A^{4-j} \frac{d^j}{dt^j} \right) \times$$

$$\times \left\{ e^{-\frac{1+i}{\sqrt{2}} tA} A^{-7/2} (CS - S + C + E)^{-1} \left[((C + iE)(S - iE) \right. \right. +$$

$$+\left(E+iC\right)\left(S+iE\right)\right)A^{1/2}\int_0^\infty e^{-\frac{1+i}{\sqrt{2}}sA}\upsilon\left(s\right)ds+$$

$$+2\left(E+iC\right)\left(S-iE\right)A^{1/2}\int_0^\infty e^{-\frac{1-i}{\sqrt{2}}sA}\upsilon\left(s\right)ds\Bigg]-$$
$$-e^{-\frac{1-i}{\sqrt{2}}tA}A^{-7/2}\left(CS-S+C+E\right)^{-1}\times$$
$$\times\Bigg[2\left(E-iC\right)\left(S+iE\right)A^{1/2}\int_0^\infty e^{-\frac{1+i}{\sqrt{2}}sA}\upsilon\left(s\right)ds+\left(\left(E-iC\right)\left(S-iE\right)+\right.\tag{12}$$
$$\left.+\left(E+iC\right)\left(E-iS\right)\right)A^{1/2}\int_0^\infty e^{-\frac{1-i}{\sqrt{2}}sA}\upsilon\left(s\right)ds\Bigg]\Bigg\}=f\left(t\right).$$

We prove that the second addend in (12) gives a completely continuous operator in $L_2\left(R_+;H\right)$. Introduce the operators

$$K_{1,j}\upsilon=A^{4-j}\frac{d^j}{dt^j}e^{-\frac{1+i}{\sqrt{2}}tA}A^{-7/2}\left(CS-S+C+E\right)^{-1}\left(\left(C+iE\right)\left(S-iE\right)+\right.$$
$$\left.+\left(E+iC\right)\left(S+iE\right)\right)A^{1/2}\int_0^\infty e^{-\frac{1+i}{\sqrt{2}}sA}\upsilon\left(s\right)ds,$$

$$K_{2,j}\upsilon=A^{4-j}\frac{d^j}{dt^j}e^{-\frac{1+i}{\sqrt{2}}tA}A^{-7/2}\left(CS-S+C+E\right)^{-1}\times$$
$$\times\left(E+iC\right)\left(S-iE\right)A^{1/2}\int_0^\infty e^{-\frac{1-i}{\sqrt{2}}sA}\upsilon\left(s\right)ds,$$

$$K_{3,j}\upsilon=A^{4-j}\frac{d^j}{dt^j}e^{-\frac{1-i}{\sqrt{2}}tA}A^{-7/2}\left(CS-S+C+E\right)^{-1}\times$$
$$\times\left(E-iC\right)\left(S+iE\right)A^{1/2}\int_0^\infty e^{-\frac{1+i}{\sqrt{2}}sA}\upsilon\left(s\right)ds,$$

$$K_{4,j}\upsilon=A^{4-j}\frac{d^j}{dt^j}e^{-\frac{1-i}{\sqrt{2}}tA}A^{-7/2}\left(CS-S+C+E\right)^{-1}\left(\left(E-iC\right)\left(S-iE\right)+\right.$$
$$\left.+\left(E+iC\right)\left(E-iS\right)\right)A^{1/2}\int_0^\infty e^{-\frac{1-i}{\sqrt{2}}sA}\upsilon\left(s\right)ds,$$

$$L_{,j}^{+}\upsilon=A^{4-j}\frac{d^j}{dt^j}\int_0^\infty e^{-\frac{1\pm i}{\sqrt{2}}(t+s)A}A^{-3}\upsilon\left(s\right)ds,\qquad j=1,2,3.$$

The boundedness of the first four operators in $L_2\left(R_+;H\right)$ follows from the inequalities

$$\left\|K_{1,j}\upsilon\right\|_{L_2}\le$$
$$\le\frac{1}{\sqrt{2}}\left\|\left(CS-S+C+E\right)^{-1}\left(\left(C+iE\right)\left(S-iE\right)+\left(E+iC\right)\left(S+iE\right)\right)\right\|\left\|\upsilon\right\|_{L_2},$$

$$\left\|K_{2,j}\upsilon\right\|_{L_2}\le\frac{1}{\sqrt{2}}\left\|\left(CS-S+C+E\right)^{-1}\left(E+iC\right)\left(S-iE\right)\right\|\left\|\upsilon\right\|_{L_2},$$

$$\|K_{3,j}v\|_{L_2} \leq \frac{1}{\sqrt{2}} \left\| (CS - S + C + E)^{-1} (E - iC)(S + iE) \right\| \|v\|_{L_2},$$

$$\|K_{4,j}v\|_{L_2} \leq \frac{1}{\sqrt{2}} \left\| (CS - S + C + E)^{-1} ((E - iC)(S - iE) + \right.$$
$$\left. + (E + iC)(E - iS)) \right\| \|v\|_{L_2}, \quad j = 1, 2, 3$$

which we obtain by application of inequalities (9) and (10). The boundednesses of the operators $L_j^{\pm}$, $j = 1, 2, 3$ in $L_2(R_+; H)$ follow from the following estimations

$$\left\| A^{4-j} \frac{d^j}{dt^j} e^{-\frac{1 \pm i}{\sqrt{2}}(t+s)A} A^{-3} \right\| \leq const\,(t + s), \quad j = 1, 2, 3.$$

Further, denote by P_m an operator of orthogonal projection on the first m eigen vectors $\{\varphi_1, ..., \varphi_m\}$ of the operator A responding to eigen values $\lambda_1, ..., \lambda_m$. Then the norms of the operators $Q_{j,m} = B_j - B_j P_m$ tend to zero as $m \to \infty$. It is obvious that the operators

$$B_j P_m K_{1,j} v = \left(-\frac{1+i}{\sqrt{2}} \right)^j \sum_{k=1}^{m} \lambda_k^{1/2} e^{-\frac{1+i}{\sqrt{2}} t \lambda_k} (K_1 v, \varphi_k) B_j \varphi_k,$$

$$B_j P_m K_{2,j} v = \left(-\frac{1+i}{\sqrt{2}} \right)^j \sum_{k=1}^{m} \lambda_k^{1/2} e^{-\frac{1+i}{\sqrt{2}} t \lambda_k} (K_2 v, \varphi_k) B_j \varphi_k,$$

$$B_j P_m K_{3,j} v = \left(-\frac{1-i}{\sqrt{2}} \right)^j \sum_{k=1}^{m} \lambda_k^{1/2} e^{-\frac{1-i}{\sqrt{2}} t \lambda_k} (K_3 v, \varphi_k) B_j \varphi_k,$$

$$B_j P_m K_{4,j} v = \left(-\frac{1-i}{\sqrt{2}} \right)^j \sum_{k=1}^{m} \lambda_k^{1/2} e^{-\frac{1-i}{\sqrt{2}} t \lambda_k} (K_4 v, \varphi_k) B_j \varphi_k, \quad j = 1, 2, 3,$$

where

$$K_1 v = (CS - S + C + E)^{-1} ((C + iE)(S - iE) + (E + iC)(S + iE)) \times$$
$$\times A^{1/2} \int_0^{\infty} e^{-\frac{1+i}{\sqrt{2}} sA} v(s)\, ds \in L(L_2(R_+; H), H),$$

$$K_2 v = (CS - S + C + E)^{-1} (E + iC)(S - iE) A^{1/2} \times$$
$$\times \int_0^{\infty} e^{-\frac{1-i}{\sqrt{2}} sA} v(s)\, ds \in L(L_2(R_+; H), H),$$

$$K_3 v = (CS - S + C + E)^{-1} (E - iC)(S + iE) A^{1/2} \times$$
$$\times \int_0^{\infty} e^{-\frac{1+i}{\sqrt{2}} sA} v(s)\, ds \in L(L_2(R_+; H), H),$$

$$K_4 v = (CS - S + C + E)^{-1} ((E - iC)(S - iE) + (E + iC)(E - iS)) \times$$
$$\times A^{1/2} \int_0^{\infty} e^{-\frac{1-i}{\sqrt{2}} sA} v(s)\, ds \in L(L_2(R_+; H), H),$$

map the space $L_2(R_+;H)$ in the finite-dimensional subspace, and the operators

$$B_jP_mL_j^{\pm}v=\left(-\frac{1\pm i}{\sqrt{2}}\right)^j\sum_{k=1}^{m}\left(\int_0^{\infty}\lambda_k e^{-\frac{1\pm i}{\sqrt{2}}(t+s)\lambda_k}(v(s),\varphi_k)\,ds\right)B_j\varphi_k,\quad j=1,2,3$$

are Hilbert-Schmidt operators in $L_2(R_+;H)$. Therefore, $B_jP_mK_{n,j}$, $B_jP_mL_j^{\pm}$, $n=1,2,3,4$, $j=1,2,3$ are completely continuous operators in $L_2(R_+;H)$.

Then it follows from the obvious inequalities

$$\|B_jK_{n,j}-B_jP_mK_{n,j}\|_{L_2\to L_2}\le\|Q_{j,m}\|\,\|K_{n,j}\|_{L_2\to L_2}\to 0,$$

$$m\to\infty,\quad j=1,2,3,\quad n=1,2,3,4,$$

$$\left\|B_jL_j^{\pm}-B_jP_mL_j^{\pm}\right\|_{L_2\to L_2}\le\|Q_{j,m}\|\left\|L_j^{\pm}\right\|_{L_2\to L_2}\to 0,\quad m\to\infty,\quad j=1,2,3$$

that the operators $B_jK_{n,j}$, $B_jL_j^{\pm}$, $n=1,2,3,4$, $j=1,2,3$ are completely continuous in $L_2(R_+;H)$. Thus, for completion of proof of the theorem it is sufficient to check normal solvability of the integral equation

$$v(t)+\frac{1}{4\sqrt{2}}\left(\sum_{j=1}^{3}B_jA^{4-j}\frac{d^j}{dt^j}\right)\times$$

$$\times\int_0^{\infty}\left((1+i)e^{-\frac{1+i}{\sqrt{2}}|t-s|A}+(1-i)e^{-\frac{1-i}{\sqrt{2}}|t-s|A}\right)A^{-3}v(s)\,ds=f(t)\qquad(13)$$

in the space $L_2(R_+;H)$. For this we introduce the notation

$$V(t)=\begin{cases}v(t), & t>0\\ v_1(-t), & t<0\end{cases}\qquad F(t)=\begin{cases}f(t), & t>0\\ f_1(-t), & t<0\end{cases}$$

and consider the following equation in $L_2(R;H)=L_2(R_+;H)+L_2(R_+;H)$

$$V(t)+\frac{1}{4\sqrt{2}}\left(\sum_{j=1}^{3}B_jA^{4-j}\frac{d^j}{dt^j}\right)\int_{-\infty}^{\infty}\left((1+i)e^{-\frac{1+i}{\sqrt{2}}|t-s|A}+(1-i)e^{-\frac{1-i}{\sqrt{2}}|t-s|A}\right)\times$$

$$\times A^{-3}V(s)\,ds=F(t),\quad t\in R$$

which is equivalent to the system of equations

$$v(t)+\frac{1}{4\sqrt{2}}\left(\sum_{j=1}^{3}B_jA^{4-j}\frac{d^j}{dt^j}\right)\times$$

$$\times\int_0^{\infty}\left((1+i)e^{-\frac{1+i}{\sqrt{2}}|t-s|A}+(1-i)e^{-\frac{1-i}{\sqrt{2}}|t-s|A}\right)\times$$

$$\times A^{-3}v(s)\,ds+\frac{1}{4\sqrt{2}}\left(\sum_{j=1}^{3}B_jA^{4-j}\frac{d^j}{dt^j}\right)\times$$

$$\times\int_0^{\infty}\left((1+i)e^{-\frac{1+i}{\sqrt{2}}(t+s)A}+(1-i)e^{-\frac{1-i}{\sqrt{2}}(t+s)A}\right)A^{-3}v_1(s)\,ds=f(t),$$

$$\upsilon_1(t)+\frac{1}{4\sqrt{2}}\left(\sum_{j=1}^{3}B_jA^{4-j}\frac{d^j}{dt^j}\right)\int_0^\infty\left((1+i)\,e^{-\frac{1+i}{\sqrt{2}}|t-s|A}+(1-i)\,e^{-\frac{1-i}{\sqrt{2}}|t-s|A}\right)\times$$

$$\times A^{-3}\upsilon_1(s)\,ds+\frac{1}{4\sqrt{2}}\left(\sum_{j=1}^{3}B_jA^{4-j}\frac{d^j}{dt^j}\right)\int_0^\infty\left((1+i)\,e^{-\frac{1+i}{\sqrt{2}}(t+s)A}+\right.$$

$$\left.+(1-i)\,e^{-\frac{1-i}{\sqrt{2}}(t+s)A}\right)A^{-3}\upsilon(s)\,ds=f_1(t)\,,$$

or in operator form

$$\begin{pmatrix} E+L & K\\ K & E+L\end{pmatrix}V=\left[\begin{pmatrix} E+L & 0\\ 0 & E+L\end{pmatrix}\begin{pmatrix} 0 & K\\ K & 0\end{pmatrix}V\right]=F.$$

It follows from the condition of the theorem that the equation

$$\begin{pmatrix} E+L & K\\ K & E+L\end{pmatrix}V=F$$

is correct and uniquely solvable at all $F\in L_2(R;H)$.

Really, after Fourier transform we have

$$\hat{V}(\xi)=P_0(-i\xi)\,P^{-1}(-i\xi)\,\hat{F}(\xi)\,.$$

Since $A^{-1}\in\sum_\infty$, $B_i\in\sum_\infty$, $i=1,2,3$ and

$$\left\|P_0(i\xi)\,P^{-1}(i\xi)\right\|_{H\to H}\le\left\|\left[E+\left(\sum_{j=1}^{3}(i\xi)^j\,B_jA^{4-j}\right)\left(\xi^4E+A^4\right)^{-1}\right]^{-1}\right\|_{H\to H},$$

then by the Keldysh lemma [5] we conclude that

$$\sup_{\xi\in R}\left\|P_0(i\xi)\,P^{-1}(i\xi)\right\|_{H\to H}\le const.$$

Therefore, $V(t)\in L_2(R;H)$. Since the operator K is completely continuous in $L_2(R_+;H)$, hence, we obtain that the operator $\begin{pmatrix} E+L & 0\\ 0 & E+L\end{pmatrix}$ has Fredholm property in the space $L_2(R;H)$, i.e. equation (13) is normally solvable in $L_2(R_+;H)$, that is equivalent to normal solvability of boundary value problem (2),(3). The theorem is proved.

The author expresses his gratitude to prof. S.S. Mirzoyev for useful discussions.

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Received March 03, 2004; Revised June 08, 2004.
Translated by Mirzoyeva K.S.