MATHEMATICS

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ON THE SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR A CLASS OF SECOND ORDER OPERATOR-DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENT AT A SECOND ORDER DERIVATIVE

Abstract

Sufficient conditions for operator coefficients of a second order operator-differential equation with discontinuous coefficient at the second order derivative are found. These conditions provide a well-posed and unique solvability of a boundary value problem for this equation.

Let H be a separable Hilbert space, A be a self-adjoined positive-definite operator in H $(A = A^* > cE,\ c > 0)$.

Let's consider an operator-differential equation of the second order

$$-\rho(t)\,u''(t) + A^2u(t) + A_1u'(t) + A_2u(t) = f(t), \quad t \in \mathbf{R}_+ = [0; +\infty), \qquad (1)$$

when it is fulfilled the boundary condition

$$u(0) = 0, \qquad (2)$$

where f$(t) \in L_2(\mathbf{R}_+, H)$, $u(t) \in W_2^2(\mathbf{R}_+; H)$ (see [1,2]), A_1 and A_2 are linear, generally speaking, unbounded operators, $\rho(t)$ is a scalar function determined by the following way:

$$\rho(t) = \begin{cases} \alpha, & if \quad 0 \le t \le 1, \\ \beta, & if \quad 1 < t < +\infty, \end{cases}$$

moreover α, β are positive, generally speaking, unequal to each other numbers.

Introduce the following denotation:

$$\mathcal{L}_0u(t) = -\rho(t)\,u''(t) + A^2u(t), \quad u(t) \in \mathring{W}_2^2(\mathbf{R}_+; H) \qquad (see \;\; [1,2]), \qquad (3)$$

$$\mathcal{L}_1u(t) = A_1u'(t) + A_2u(t), \quad u(t) \in \mathring{W}_2^2(\mathbf{R}_+; H),$$

$$\mathcal{L}u(t) = \mathcal{L}_0u(t) + \mathcal{L}_1u(t), \quad u(t) \in \mathring{W}_2^2(\mathbf{R}_+; H).$$

In the paper we find sufficient conditions on the coefficients of operator-differential equation (1), providing well-posed and univalent solvability of boundary-value problem (1), (2). Here, the estimations of norms of intermediate derivatives are obtained through the principal part of equation (1) in the subspace $\mathring{W}_2^2(\mathbf{R}_+; H)$, that is of mathematical interest. Note that in the papers [1,2] it is studied boundary value problem (1), (2) with discontinuous coefficients, not at the second order derivative, but at $A^2 u(t)$. In the paper [3] an asymptotics of eigen-values of some boundary-value problem is studied for the equation $\mathcal{L}_0 u(t) - \lambda u(t) = 0$ on a finite segment with conjugation conditions at the discontinuity point of the coefficient $\rho(t)$.

Theorem 1. *The operator $\mathcal{L}_0$ determined by equality (3) realizes isomorphism between the spaces $\overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$ and $L_2(\mathbf{R}_+;H)$.*

Proof. Obviously, a homogeneous equation $\mathcal{L}_0 u(t) = 0$ has only zero solution from the space $\overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$. This follows from the fact that the solution of the equation $\mathcal{L}_0 u(t) = 0$ from $\overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$ is of the form:

$$\tilde{u}(t) = \begin{cases} \tilde{u}_1(t) = e^{-\frac{t}{\sqrt{\alpha}}A}\tilde{\varphi}_0 + e^{-\frac{1-t}{\sqrt{\alpha}}A}\tilde{\varphi}_1, & if \quad 0 \le t < 1, \\ \tilde{u}_2(t) = e^{-\frac{t-1}{\sqrt{\beta}}A}\tilde{\varphi}_2, & if \quad 1 < t < +\infty, \end{cases}$$

where the vectors $\tilde{\varphi}_j \in D\left(A^{3/2}\right)$, $j = 0,1,2$ (see, for example [4]) are the desired elements of the space H.

To define these elements, from the conditions $\tilde{u}(t) \in \overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$ we get the following relations:

$$\begin{cases} \tilde{u}(0) = \tilde{u}_1(0) = 0, \\ \tilde{u}(1) = \tilde{u}_1(1) = \tilde{u}_2(1), \\ \tilde{u}'(1) = \tilde{u}_1'(1) = \tilde{u}_2'(1), \end{cases}$$

from which all $\tilde{\varphi}_j = 0$, $j = 0,1,2$, i.e. $\tilde{u}(t) = 0$, are easily obtained. Now, show that at any $f(t) \in L_2(\mathbf{R}_+;H)$ there exists $u(t) \in \overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$ for which $\mathcal{L}_0 u(t) = f(t)$. In the space $W_2^2(\mathbf{R};H)$ $(\mathbf{R} = (-\infty;+\infty))$ (see [4]) let's consider the equation

$$\mathcal{L}_\alpha v(t) \equiv -\alpha v''(t) + A^2 v(t) = F(t), \tag{4}$$

where

$$F(t) = \begin{cases} f(t), & if \quad t \in [0;1), \\ 0, & if \quad t \in \mathbf{R} \backslash [\mathbf{0};\mathbf{1}). \end{cases}$$

It is easily seen that

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\alpha\lambda^2 E + A^2\right)^{-1}\left(\int_0^1 f(s)\, e^{-i\lambda s} ds\right) e^{i\lambda t} d\lambda, \quad t \in \mathbf{R}$$

belongs to the space $W_2^2(\mathbf{R};H)$ and satisfies equation (4). Now define the contraction of the solution $v(t)$ on $[0;1)$ and denote it by $u_\alpha(t)$.

Similarly we consider the equation

$$\mathcal{L}_\beta v(t) \equiv -\beta v''(t) + A^2 v(t) = F(t), \tag{5}$$

where

$$F(t) = \begin{cases} f(t), & if \quad t \in (1;+\infty), \\ 0, & if \quad t \in \mathbf{R} \backslash (1;+\infty) \end{cases}$$

and define the solution $u_\beta(t)$ of equation (5) from the space $W_2^2((1;+\infty);H)$ (see [4]).

So, the solution of the equation $\mathcal{L}_0 u(t) = f(t)$ from the space $\overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$ is represented in the form

$$u(t) = \begin{cases} u_1(t) = u_\alpha(t) + e^{-\frac{t}{\sqrt{\alpha}}A}\varphi_0 + e^{-\frac{1-t}{\sqrt{\alpha}}A}\varphi_1, & if \quad 0 \le t < 1, \\ u_2(t) = u_\beta(t) + e^{-\frac{t-1}{\sqrt{\beta}}A}\varphi_2, & if \quad 1 < t < +\infty, \end{cases}$$

where vectors $\varphi_j \in D(A^{3/2})$, $j = 0,1,2$ are the elements from H, that are uniquely determined from the condition $u(t) \in \overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$ by the following relations:

$$\begin{cases} u(0) = u_1(0) = 0, \\ u(1) = u_1(1) = u_2(1), \\ u'(1) = u_1'(1) = u_2'(1). \end{cases}$$

On the other hand, the operator $\mathcal{L}_0 : \overset{\circ}{W}{}_2^2(\mathbf{R}_+;H) \to L_2(\mathbf{R}_+;H)$ is continuous.

Then, taking into account Banach inverse operator theorem we have that $\mathcal{L}_0 : \overset{\circ}{W}{}_2^2(\mathbf{R}_+;H) \to L_2(\mathbf{R}_+;H)$ is an isomorphism. The theorem is proved.

It follows from this theorem that $\|\mathcal{L}_0 u\|_{L_2(\mathbf{R}_+;H)}$ is norm in the space $\overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$ which is equivalent to the initial norm $\|u\|_{W_2^2(\mathbf{R}_+;H)}$ (see [1,2]). Now, let's study boundary value problem (1), (2). It holds the following conditional theorem.

Theorem 2. *Let $A = A^* > cE$, $c > 0$, the operators $A_j A^{-j}$, $j = 1,2$ be bounded in H and the inequality*

$$\gamma_1 \left\|A_1 A^{-1}\right\| + \gamma_2 \left\|A_2 A^{-2}\right\| < 1,$$

where $\gamma_j \in (0;+\infty)$, $j = 1,2$, be fulfilled.

Then boundary-value problem (1), (2) at any $f(t) \in L_2(\mathbf{R}_+;H)$ has a unique solution from $W_2^2(\mathbf{R}_+;H)$.

Proof. Write boundary value problem (1), (2) in the form of operator equation $(\mathcal{L}_0 + \mathcal{L}_1)u(t) = f(t)$, where $f(t) \in L_2(\mathbf{R}_+;H)$, $u(t) \in \overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$. As the operator $\mathcal{L}_0$ by theorem 1 has the bounded inverse $\mathcal{L}_0^{-1}$ acting from $L_2(\mathbf{R}_+;H)$ on $\overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$, then by substituting $u(t) = \mathcal{L}_0^{-1}v(t)$ we get the following equation in $L_2(\mathbf{R}_+;H)$:

$$\left(E + \mathcal{L}_1\mathcal{L}_0^{-1}\right)v(t) = f(t).$$

On the other hand

$$\left\|\mathcal{L}_1\mathcal{L}_0^{-1}v\right\|_{L_2(\mathbf{R}_+;H)} = \|\mathcal{L}_1 u\|_{L_2(\mathbf{R}_+;H)} \le$$

$$\le \left\|A_1A^{-1}\right\| \left\|Au'\right\|_{L_2(\mathbf{R}_+;H)} + \left\|A_2A^{-2}\right\| \left\|A^2u\right\|_{L_2(\mathbf{R}_+;H)}.$$

Here, applying the theorem on intermediate derivatives (see [4]), we have:

$$\left\|\mathcal{L}_1\mathcal{L}_0^{-1}v\right\|_{L_2(\mathbf{R}_+;H)} \le \gamma_1 \left\|A_1A^{-1}\right\| \|\mathcal{L}_0 u\|_{L_2(\mathbf{R}_+;H)} +$$

$$+\gamma_2 \left\| A_2 A^{-2} \right\| \left\| \mathcal{L}_0 u \right\|_{L_2(\mathbf{R}_+;H)} = \left(\gamma_1 \left\| A_1 A^{-1} \right\| + \gamma_2 \left\| A_2 A^{-2} \right\| \right) \left\| v \right\|_{L_2(\mathbf{R}_+;H)} .$$

Therefore, by fulfilling the inequality $\gamma_1 \left\| A_1 A^{-1} \right\| + \gamma_2 \left\| A_2 A^{-2} \right\| < 1$ the operator $E + \mathcal{L}_1 \mathcal{L}_0^{-1}$ is invertible and we can find $u(t)$. The theorem is proved.

Here, there arises a problem on precise estimation of norms of intermediate derivative operators by $\left\| \mathcal{L}_0 u \right\|_{L_2(\mathbf{R}_+;H)}$ that also is of independent mathematical interest.

Theorem 3. *Let* $u(t) \in \overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$. *Then the following inequalities are valid*

$$\left\| A u' \right\|_{L_2(\mathbf{R}_+;H)} \le \gamma_1 \left\| \mathcal{L}_0 u \right\|_{L_2(\mathbf{R}_+;H)} ,$$

$$\left\| A^2 u \right\|_{L_2(\mathbf{R}_+;H)} \le \gamma_2 \left\| \mathcal{L}_0 u \right\|_{L_2(\mathbf{R}_+;H)} ,$$

where

$$\gamma_1 = \frac{1}{2 \min^{1/2}(\alpha;\beta)}, \quad \gamma_2 = \frac{\max^{1/2}(\alpha;\beta)}{\min^{1/2}(\alpha;\beta)} .$$

Proof. As

$$\mathcal{L}_0 u(t) = -\rho(t) u''(t) + A^2 u(t) = f(t),$$

then, multiplying scalarly in the space $L_2(\mathbf{R}_+;H)$ the both sides of this equation by $\rho^{-1}(t) A^2 u(t)$:

$$\left(\mathcal{L}_0 u, \rho^{-1}(t) A^2 u \right)_{L_2(\mathbf{R}_+;H)} = \left(-\rho(t) u'' + A^2 u, \rho^{-1}(t) A^2 u \right)_{L_2(\mathbf{R}_+;H)} =$$

$$= \left(-\rho(t) u'', \rho^{-1}(t) A^2 u \right)_{L_2(\mathbf{R}_+;H)} + \left(A^2 u, \rho^{-1}(t) A^2 u \right)_{L_2(\mathbf{R}_+;H)} =$$

$$= -\left(u'', A^2 u \right)_{L_2(\mathbf{R}_+;H)} + \left\| \rho^{-1/2}(t) A^2 u \right\|^2_{L_2(\mathbf{R}_+;H)}$$

Since, at $u(t) \in \overset{\circ}{W}{}_2^2(\mathbf{R}_+;H)$

$$\left(\mathcal{L}_0 u, \rho^{-1}(t) A^2 u \right)_{L_2(\mathbf{R}_+;H)} = \left\| A u' \right\|^2_{L_2(\mathbf{R}_+;H)} +$$

$$+ \left\| \rho^{-1/2}(t) A^2 u \right\|^2_{L_2(\mathbf{R}_+;H)} \ge \left\| \rho^{-1/2}(t) A^2 u \right\|^2_{L_2(\mathbf{R}_+;H)}, \qquad (6)$$

then, at first applying to the left hand side of (6) Bunyakovskii-Schwatz inequality and then Young inequality we get:

$$\left| \left(\mathcal{L}_0 u, \rho^{-1}(t) A^2 u \right)_{L_2(\mathbf{R}_+;H)} \right| \le \left\| \mathcal{L}_0 u \right\|_{L_2(\mathbf{R}_+;H)} \left\| \rho^{-1}(t) A^2 u \right\|_{L_2(\mathbf{R}_+;H)} \le$$

$$\le \frac{1}{\min^{1/2}(\alpha;\beta)} \left\| \mathcal{L}_0 u \right\|_{L_2(\mathbf{R}_+;H)} \left\| \rho^{-1/2}(t) A^2 u \right\|_{L_2(\mathbf{R}_+;H)} \le$$

$$\le \frac{\varepsilon}{2} \frac{1}{\min(\alpha;\beta)} \left\| \mathcal{L}_0 u \right\|^2_{L_2(\mathbf{R}_+;H)} + \frac{1}{2\varepsilon} \left\| \rho^{-1/2}(t) A^2 u \right\|^2_{L_2(\mathbf{R}_+;H)}, \quad (\varepsilon > 0). \qquad (7)$$

Now, choosing $\varepsilon = \frac{1}{2}$ in inequality (7) allowing for (6) we get:

$$\left\| A u' \right\|^2_{L_2(\mathbf{R}_+;H)} \le \frac{1}{4 \min(\alpha;\beta)} \left\| \mathcal{L}_0 u \right\|^2_{L_2(\mathbf{R}_+;H)} = \gamma_1^2 \left\| \mathcal{L}_0 u \right\|^2_{L_2(\mathbf{R}_+;H)} .$$

On the other hand, from (6) allowing for (7) we have:

$$\left\|\rho^{-1/2}(t)\,A^2u\right\|^2_{L_2(\mathbf{R}_+;H)}\le$$

$$\le\frac{1}{\min^{1/2}(\alpha;\beta)}\left\|\mathcal{L}_0u\right\|_{L_2(\mathbf{R}_+;H)}\left\|\rho^{-1/2}(t)\,A^2u\right\|_{L_2(\mathbf{R}_+;H)},$$

hence, it follows

$$\frac{1}{\min^{1/2}(\alpha;\beta)}\left\|\mathcal{L}_0u\right\|_{L_2(\mathbf{R}_+;H)}\ge\left\|\rho^{-1/2}(t)\,A^2u\right\|_{L_2(\mathbf{R}_+;H)}\ge$$

$$\ge\frac{1}{\max^{1/2}(\alpha;\beta)}\left\|A^2u\right\|_{L_2(\mathbf{R}_+;H)}.$$

Thereby we finally get

$$\left\|A^2u\right\|_{L_2(\mathbf{R}_+;H)}\le\frac{\max^{1/2}(\alpha;\beta)}{\min^{1/2}(\alpha;\beta)}\left\|\mathcal{L}_0u\right\|_{L_2(\mathbf{R}_+;H)}=\gamma_2\left\|\mathcal{L}_0u\right\|_{L_2(\mathbf{R}_+;H)}.$$

The theorem is proved.

Finally, using the numbers $\gamma_j,\ j=1,2$ from theorem 3, we formulate the exact statement of the theorem on well-posed and unique solvability of boundary value problem (1), (2).

Theorem 4. *Let $A=A^*>cE$, $c>0$, the operators A_jA^{-j}, $j=1,2$ be bounded in H and the inequality*

$$\frac{1}{2\min^{1/2}(\alpha;\beta)}\left\|A_1A^{-1}\right\|+\frac{\max^{1/2}(\alpha;\beta)}{\min^{1/2}(\alpha;\beta)}\left\|A_2A^{-2}\right\|<1\,.$$

be fulfilled.

Then boundary-value problem (1), (2) at any $f(t)\in L_2(\mathbf{R}_+;H)$ has a unique solution from $W_2^2(R_+;H)$.

Remark. Note that we can study boundary value problem (1), (2) in a corresponding way in case when $\rho(t)$ is any positive fucntion having a finite number of discontinuity points of the first genus.

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