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AFFINOR STRUCTURES ON A MANIFOLD WITH PURE RIEMANN METRIC

Abstract

In this paper the condition for almost integrable affinor structure on Rieman connection manifold is determined. The conditions for integrable affinor structure on Riemann monifold with pure metric are found.

Let the Riemann metrics g (it need not be positive defined) be given on differentiable manifold M_n of the class C^∞ provided by nonintegrable affinor φ-structure. This metric is called pure with respect to φ structure if for all vector fields X and Y the condition

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{1}$$

is statisfied.

The manifold M_n with the pure Riemann metric is called B-manifold if φ structure is integrable and the metric is holomorphic (the purity of partial derivatives of the pure tensor g on admissible local coordinates). The papers of V.V.Vishnevsky and his followers (see [1], [2]) are dedicated to the investigation of geometry of $B-$ manifolds.

1. The Tachibana operator on M_n is determined by the following form (see [7], [8]).

$$(\Phi\omega)(x, z_1, z_2) = (L_{\varphi x}\omega)(z_1, z_2) - (L_x(\omega \circ \varphi))(z_1, z_2) +$$

$$+\omega(z_1, \varphi(L_x z_2)) - \omega(\varphi z_1, (L_x z_2)),\ \ x, z_1, z_2 \in T_0^1(M_n), \tag{2}$$

where L_x is Lie differentiation, $\omega(z_1, z_2)$ is an arbitrary pure tensor of type $(0, 2)$ with respect to $\varphi-$ structure

$$\omega(\varphi z_1, z_2) = \omega(z_1, \varphi z_2),\ \ (\omega \circ \varphi)\,(z_1, z_2) \overset{def}{=} \omega\,(\varphi z_1, z_2) = \omega(z_1, \varphi z_2).$$

Note that the Tachibana operator determines the tensor of type $(0, 3)$. With the help of this operator the new approach is realized in theory of complete lifts of structures (see [4], [9]).

Let ∇ be an operator of covariant differentiation in some linear connection Γ. The Vishnevsky [1] operator on M_n is determined as follows

$$(\psi\omega)\,(x, z_1, z_2,) = (\nabla_{\varphi x}\omega)\,(z_1, z_2) - (\nabla_x(\omega \circ \varphi))\,(z_1, z_2) =$$

$$= (\nabla_{\varphi x}\omega)(z_1, z_2) - (\nabla_x\omega)\,(\varphi z_1, z_2)\,. \tag{3}$$

Operator (3) is applied to investigation of holomorphy of the tensor field ω (see [3, p.184]). This operator is interesting so that this plays an important part in determination of horizontal lifts of a structure (see [4]).

Theorem 1. *If the $\varphi-$ structure on M_n is almost integrable in connection Γ, then Tachibana and Vishnevsky operators coincide.*

Proof. From the formulae of tensor torsion $T(X,Y)$ of the connection Γ we have

$$L_x Y = [X,Y] = \nabla_x Y - \nabla_y X - T(X,Y). \tag{4}$$

Using (2), (4) and formulae (see [6, p.41])

$$(L_x\omega)(Y_1,Y_2) = X(\omega(Y_1,Y_2)) - \omega([X,Y_1],Y_2) - \omega(Y_1,[X,Y_2])$$

we obtain

$$(\Phi\omega)(x,z_1,z_2) = \varphi(x)(\omega(z_1,z_2)) - \omega(\nabla_{\varphi x}z_1 - \nabla_{z_1}\varphi(x) - T(\varphi x,z_1),z_2) -$$
$$-\omega(z_1,\nabla_{\varphi x}z_2 - \nabla_{z_2}\varphi(x) - T(\varphi x,z_2)) - x(\omega(\varphi z_1,z_2)) +$$
$$+(\omega\circ\varphi)(\nabla_x z_1 - \nabla_{z_1}x - T(x,z_1),z_2) + (\omega\circ\varphi)(z_1,\nabla_x z_2 - \nabla_{z_2}x - T(x,z_2)) +$$
$$+\omega(z_1,\varphi(\nabla_x z_2 - \nabla_{z_2}x - T(x,z_2))) - \omega(\varphi z_1,\nabla_x z_2 - \nabla_{z_2}x - T(x,z_2)).$$

Allowing for the conditions of purity ω we found

$$(\Phi\omega)(x,z_1,z_2) = \varphi(x)(\omega(z_1,z_2)) - x(\omega(\varphi z_1,z_2)) -$$
$$-\omega(\nabla_{\varphi x}z_1,z_2) + \omega(\nabla_{z_1}\varphi(x),z_2) + \omega(T(\varphi x,z_1),z_2) -$$
$$-\omega(z_1,\nabla_{\varphi x}z_2) + \omega(z_1,\nabla_{z_2}\varphi(x)) + \omega(z_1,T(\varphi x,z_2)) +$$
$$+\omega(\varphi(\nabla_x z_1),z_2) - \omega(\varphi(\nabla_{z_1}x),z_2) - \omega(\varphi(T(x,z_1)),z_2) +$$
$$+\omega(\varphi z_1,\nabla_x z_2) - \omega(z_1,\varphi(\nabla_{z_2}x)) - \omega(z_1,\varphi(T(x,z_2))). \tag{5}$$

By application of the formulae (see [6, p.22])

$$(\nabla K)(x_1,...,x_S,x) = (\nabla_x K)(x_1,...,x_S) = \nabla_x(K(x_1,...,x_S)) -$$
$$-\sum_{i=1}^{S} K(x_1,...,\nabla_x x_i,...,x_S) \tag{6}$$

we obtain

$$\omega(\nabla_{z_1}\varphi(x),z_2) - \omega(\varphi(\nabla_{z_1}x),z_2) + \omega(z_1,\nabla_{z_2}\varphi(x)) -$$
$$-\omega(z_1,\varphi(\nabla z_2 x)) = \omega((\nabla\varphi)(x,z_1),z_2) + \omega(z_1,(\nabla\varphi)(x,z_2)). \tag{7}$$

From (5) and (7) we have

$$(\Phi\omega)(x,z_1,z_2)=\varphi(x)(\omega(z_1,z_2))-x(\omega(\varphi z_1,z_2))-\omega(\nabla_{\varphi x}z_1,z_2)+$$

$$+\omega((\nabla\varphi)(x,z_1),z_2)+\omega(z_1,(\nabla\varphi)(x,z_2))+\omega(T(\varphi x,z_1),z_2)-\omega(z_1,\nabla_{\varphi x}z_2)+$$

$$+\omega(z_1,T(\varphi x,z_2))+\omega(\varphi(\nabla_x z_1),z_2)-\omega(\varphi(T(x,z_1)),z_2)+\omega(\varphi z_1,\nabla_x z_2)-$$

$$-\omega(z_1,\varphi(T(x,z_2))). \tag{8}$$

Hence by virtue of almost integrability of φ structure ($\nabla\varphi=0, T=0$) we obtain

$$(\Phi\omega)(x,z_1,z_2)=\varphi(x)(\omega(z_1,z_2))-x(\omega(\varphi z_1,z_2))-\omega(\nabla_{\varphi x}z_1,z_2))-$$

$$-\omega(z_1,\nabla_{\varphi x}z_2)+\omega(\varphi(\nabla_x z_1),z_2)+\omega(\varphi z_1,\nabla_x z_2),$$

which subject to (6) and $\nabla_x f=Xf$ is a Vishnevsky operator.

The theorem is proved.

2. The following one holds.

Theorem 2. *In order that φ structure on M_n be almost integrable in Riemann connection Γ, it is necessary and sufficient that*

$$(\Phi g)(x,z_1,z_2)=0 \tag{9}$$

for all vector fields x, z_1 and z_2.

Proof. The necessit of the conditions at once follows from theorem 1 and condition $(\nabla_x g)(z_1,z_2)=0$.

Sufficiency. *Let condition (9) hold. Bearing in mind that for Riemann connection the torsion $T(X,Y)=0$ and*

$$\varphi(x)(g(z_1,z_2))-g(\nabla_{\varphi x}z_1,z_2)-g(z_1,\nabla_{\varphi x}z_2)=(\nabla_{\varphi x})(z_1,z_2)=0,$$

then from (8) it is evident that on M_n the Tachibana operator will have the form

$$(\Phi g)(x,z_1,z_2)=-X(g(\varphi z_1,z_2))+g((\nabla_{z_1}\varphi)(x),z_2)+$$

$$+g(z_1,(\nabla_{z_2}\varphi)(x))+g(\varphi(\nabla_x z_1),z_2)+g(\varphi z_1,\nabla_x z_2). \tag{10}$$

On the other hand from the condition

$$X(g(\varphi z_1,z_2))-g(\nabla_x\varphi(z_1),z_2))-g(\varphi z_1,\nabla_x z_2)=(\nabla_x g)(\varphi z_1,z_2)=0$$

we obtain

$$-X(g(\varphi z_1,z_2))+g(\varphi z_1,\nabla_x z_2)=-g(\nabla_x\varphi(z_1),z_2). \tag{11}$$

Substituting (11) in (10) we have

$$(\Phi g)(x, z_1, z_2) = -g(\nabla_x \varphi(z_1), z_2) + g(\varphi(\nabla_x z_1), z_2)+$$

$$+g((\nabla_{z_1}\varphi)(x), z_2) + g(z_1, (\nabla_{z_2}\varphi)(x)). \tag{12}$$

Two members from the right subject to (6) may be written as:

$$g(\varphi(\nabla_x z_1), z_2) - g(\nabla_x \varphi(z_1), z_2) = g(\varphi(\nabla_x z_1) - \nabla_x \varphi(z_1), z_2) =$$

$$= -g((\nabla_x \varphi)(z_1), z_2). \tag{13}$$

From (12) and (13) we obtain

$$(\Phi g)(x, z_1, z_2) = -g((\nabla_x \varphi)(z_1), z_2) + g((\nabla_{z_1}\varphi)(x), z_2) + g(z_1, (\nabla_{z_2}\varphi)(x)). \tag{14}$$

Analogously we obtain

$$(\Phi g)(z_2, z_1, x) = -g((\nabla_{z_2}\varphi)(z_1), x) + g((\nabla_{z_1}\varphi)(z_2), x) + g(z_1, (\nabla_x \varphi)(z_2)). \tag{15}$$

From the fact that the metrics tensor is pure it follows,

$$g((\nabla_y \varphi)(z), x) = g(z, (\nabla_y \varphi)(x)). \tag{16}$$

Really, from (1) we obtain

$$Yg(z, \varphi x) = Yg(\varphi z, x)$$

or

$$(\nabla_y g)(z, \varphi x) + g(\nabla_y z, \varphi x) + g(z, \nabla_y \varphi(x)) = (\nabla_y g)(\varphi z, x) +$$
$$+g(\nabla_y \varphi(z), x) + g(\varphi z, \nabla_y x).$$

Then

$$g(\nabla_y z, \varphi x) - g(\nabla_y \varphi(z), x) = g(\varphi z, \nabla_y x) - g(z, \nabla_y \varphi(x)).$$

Hence, using (1) and (6) we obtain (16).

From (14), (15), (16) we obtain

$$(\Phi g)(x, z_1, z_2) + (\Phi g)(z_2, z_1, x) = g((\nabla_{z_1}\varphi)(x), z_2) + g((\nabla_{z_1}\varphi)(z_2), x) =$$

$$= g((\nabla_{z_1}\varphi)(x), z_2) + g(z_2, \nabla_{z_1}\varphi)(x)) = 2g((\nabla_{z_1}\varphi)(x), z_2). \tag{17}$$

By non-degeneracy of the tensor g from (17) it follows that under the condition (9) $\varphi-$ structure in a Riemann connection satisfies the condition $\nabla\varphi = 0$. The sufficiency is proved.

Corollary 1. Let M_n be a Riemann manifold with a pure metric. If $\varphi-$ structure on M_n satisfies condition (9), then this will be integrable.

Note that in a lot of papers (see [7], [8]) the condition of form (9) is called g condition of almost holomorphy of tensor field. In natural frame $\{\partial_i\}$ it has the form

$$\varphi_k^m \partial_m g_{ji} - \partial_k \left(g_{mi}\varphi_j^m\right) + g_{jm}\partial_i\varphi_k^m + g_{mi}\partial_j\varphi_k^m = 0.$$

In [5] it is proved that the integrable $\varphi-$ structure on M_n with pure Riemann metric is covariance constantly in Riemann connection if and only if the metric is holomorphic. Subject to this fact we obtain that the following one is valid.

Corollary. The Riemann manifold with pure metric is $B-$ manifold if and only if condition (9) holds.

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