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HÖLDER WEIGHT ESTIMATES OF SINGULAR INTEGRALS GENERATED BY GENERALIZED SHIFT OPERATOR

Abstract

Systematic investigations of multidimentional singular integrals generated by generalized shift operator begin from the papers [1,2], where for these integrals, Privalov type theorems were proved. In the given paper these integrals are studied in Hölder weight spaces $H^{\gamma}_{\alpha\beta}$ ([3]). Sufficient conditions for α,β,γ providing their invariance, were found.

Note that these conditions are terminal in the operators with characteristics satisfying the Hölder condition with index $\delta>\gamma$ ([4]).

1. Singular integral generated by a generalized shift operator.

Let $\mathbf{R}_m$ be Euclidean space of dimension m $(m\geq 2)$, $R^+_m=\{(x_1,...,x_{m-1},x_m)\in\in\mathbf{R}_m:\ x_m>0\}$, $s^+_m=\{x\in R^+_m:|x|=1\}$, T^Y be a generalized shift operator (briefly GS0) ([5]), which acts according to the law

$$T^S u(x)=C_v\int_0^{\pi}u\left(x'-s';\ \sqrt{x_m^2-2x_ms_m\cos\alpha+s_m^2}\right)\sin^{2\nu-1}\alpha d\alpha, \qquad (1)$$

where $\nu>0$ $x=(x',x_m)$, $s=(s',s_m)$, $x',s'\in R_{m-1}$, $c_\nu=\Gamma\left(\nu+\frac{1}{2}\right)/\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)$. It is known that this shift is closely connected with the Bessel differential operator

$$B_{x_m}=\frac{\partial^2}{\partial x_m^2}+\frac{2\nu}{x_m}\frac{\partial}{\partial x_m}\ .$$

The singular integral

$$Au(x)=V.p.\int_{\mathbf{R}^+_m}\frac{f(\theta)}{|s|^{m+2\nu}}\left[T^S u(x)\right]s_m^{2\nu}ds=\lim_{\varepsilon\to+0}A_\varepsilon u(x), \qquad (2)$$

where

$$A_\varepsilon u(x)=\int_{\left\{s\in\mathbf{R}^+_m:|s|>\varepsilon\right\}}\frac{f(\theta)}{|s|^{m+2\nu}}\left[T^S u(x)\right]s_m^{2\nu}ds,\ \ \theta=s/|s|,\quad \varepsilon>0,$$

is called a singular integral (briefly SI) generated by GSO T^S ([1]).

Lemma 1 ([1]). *Let a and b be arbitrary numbers such that $0<a<b\leq+\infty$. Then for any point $x\in\mathbf{R}^+_m$ the following equality holds*

$$\int_{\left\{s\in\mathbf{R}^+_m:\ a<|s|<b\right\}}f(s/|s|)\,|s|^{-m-2\nu}\left[T^S u(x)\right]s_m^{2\nu}ds=$$
$$=\frac{1}{2}c_\nu\int_{\{y\in\mathbf{R}_{m+1}:\ a<|\tilde{x}-y|<b\}}f\left(\tilde{\theta}\right)|\tilde{x}-y|^{-m-2\nu}u\left(y';\sqrt{y_m^2+y_{m+1}^2}\right)|y_{m+1}|^{2\nu-1}dy, \qquad (3)$$

where $\tilde{x} = (x', x_m, 0)$, $y = (y', y_m, y_{m+1})$, $dy = dy_1 ... dy_{m+1}$,

$$\tilde{\theta} = \left(\frac{x' - y'}{r_{\tilde{x}y}}; \frac{\sqrt{(x_m - y_m)^2 + y_{m+1}^2}}{r_{\tilde{x}y}} \right), \quad r_{\tilde{x}y} = |\tilde{x} - y| \ .$$

Proof. Denote by $\tilde{A}_{a,b}(x)$ the integral in the right-hand side. Making change of variables $\tilde{x} - y = z \equiv (z'; z_m, z_{m+1})$, we obtain

$$A_{ab} = \int\limits_{\{z \in \mathbf{R}_{m+1}:\ a<|z|<b\}} f\left(\frac{z'}{|z|}; \frac{\sqrt{z_m^2 + z_{m+1}^2}}{|z|} \right) \times$$

$$\times |z|^{-m-2v} u \left(x' - z'; \sqrt{(x_m - z_m)^2 + z_{m+1}^2} \right) |z_{m+1}|^{2\nu - 1} dz \ .$$

Let us pass to new variables $(s'; s_{m-1}, s_m)$

$$\alpha : z' = s', \ z_m = s_m \cos\alpha, \ |z_{m+1}| = s_m \sin\alpha \quad (0 \le \alpha < \pi \quad \textit{and} \quad s_m > 0) \, .$$

Taking into account that the Jacobian of the transformation is s_m, we obtain

$$\frac{1}{2} C_\nu \tilde{A}_{ab}(x) = \int\limits_{\{s \in \mathbf{R}_m^+:\ a<|s|<b\}} f\left(\frac{s}{|s|}\right) |s|^{-m-2\nu} \times$$

$$\times \left(C_\nu \int\limits_0^\pi u \left(x' - s'; \sqrt{x_m^2 - 2x_m s_m \cos\alpha + s_m^2} \right) (s_m \sin\alpha)^{2\nu - 1} d\alpha \right) \times$$

$$\times s_m ds = \int\limits_{\{s \in \mathbf{R}_m^+:\ a<|s|<b\}} f\left(\frac{s}{|s|}\right) |s|^{-m-2\nu} \left[T^s u(x) \right] s_m^{2\nu} ds \ .$$

Equality (3) and also lemma 1 are proved.

If we assume in (3) $u(x) \equiv 1$ (then $T^S u(x) \equiv 1$) and pass to the polar coordinates, we obtain

$$\int\limits_{S_m^+} f(\theta) \theta_m^{2\nu} dS(\theta) = \tfrac{1}{2} C_\nu \int\limits_{S_{m+1}} f\left(\tilde{\theta}\right) |\theta_{m+1}|^{2\nu - 1} dS\left(\tilde{\theta}\right), \tag{4}$$

$$S_{m+1} = \{ y \in R_{m+1} :\ |y| = 1 \} \ .$$

Later on "C" is a constant; its exact value is not essential for us; $a(x) \prec b(x)$ means that $a(x) \le cb(x)$, where c doesn't depend on x.

In the case when $a(x) \prec b(x)$ and $b(x) \prec a(x)$ we will write $a(x) \overset{\cup}{\underset{\cap}{}} b(x)$.

2. Hölder space with weight $\mathbf{H}_{\alpha\beta}^\gamma (R_m^+)$.

Let $\gamma > 0$, $\alpha > 0$, β be a real number,

$$\rho(x) = x_m^\alpha (1 + |x|)^{\beta - \alpha} \qquad x \in \mathbf{R}_m^+ \ .$$

By definition ([3]) $u \in \mathbf{H}^{\gamma}_{\alpha\beta}(R^{+}_{m})$ if

$$\lim_{x\to\infty} u(x)\,\rho(x) = 0, \quad \lim_{x_m\to 0} u(x)\,\rho(x) = 0,$$

and the norm

$$\|u\|_{H^{\gamma}_{\alpha\beta}} = \sup_{x,y\in\mathbf{R}^{+}_{m}} \left(|u(x)\,\rho(x) - u(y)\,\rho(y)|\, d^{-\gamma}(x,y)\right) \ ,$$

is finite, where

$$d(x,y) = |x-y|\,(1+|x|)\,(1+|y|)^{-1} \ .$$

If the contrary is not stipulated, then later on we will assume that

$$0<\gamma<1, \quad 0<\alpha-\gamma<1, \quad 0<\beta+\gamma<m. \tag{5}$$

Let $x \in R^{+}_{m}$. Denote

$$\omega_x = \left\{s \in \mathbf{R}^{+}_{m} : |s-x| < \frac{x_m}{2}\right\}, \quad \omega'_x = \left\{y \in R_{m+1} : |\tilde{x}-y| < \frac{x_m}{2}\right\},$$
$$l = 2v+\beta-\alpha, \ \Psi_{\gamma}(x) = x_m^{\gamma-\alpha}(1+|x|)^{-l} \equiv \rho^{-1}(x)\left(x_m(1+|x|)^{-2}\right)^{\gamma}.$$

Remark 1. Note that if $y \in \omega'_x$, then $|x| < 2|y| < 3|x|$;

$$\omega'_x \subset \omega_x \times \{y_{m+1}\} < \frac{x_m}{2} \qquad \textit{and} \quad \textit{at} \ \ y \in R_{m+1}\backslash\omega'_x$$

$$|\tilde{x}-y| \underset{\cap}{\overset{\cup}{}} |x'-y'| + x_m + |y_m| + |y_{m+1}| \ .$$

The spaces $H^{\gamma}_{\alpha\beta}$ can be determined in terms of inequalities. The following lemma is valid.

Lemma 2. ([4]). *Let* $0<\gamma<\alpha, \ \ \beta+\gamma>0$. $u \in H^{\gamma}_{\alpha\beta}$ *iff*

$$a) \ \exists C_1(u), \ \ \forall x \in R^{+}_{m}, \ \ |u(x)| \le C_1(u)\,\Psi_{\gamma}(x),$$
$$b) \ \exists C_2(u), \ \ \forall x \in R^{+}_{m}, \ \forall y \in \omega'_x,$$

$$|u(x) - u(y)| \le C_2(u)\,\rho^{-1}(x)\, d^{\gamma}(x,y) \ .$$

Moreover,

$$(\min C_1(u) + \min C_2(u)) \underset{\cap}{\overset{\cup}{}} \|u\| \ .$$

We cite the important corollary to this lemma.

Corollary 1. *If* $u \in H^{\gamma}_{\alpha\beta}$, *then*

$$a) \quad \left|u\left(y', \sqrt{y_m^2 + y_{m+1}^2}\right)\right| \prec \|u\| \ \ (|y_m| + |y_{m+1}|)^{\gamma-\alpha}(1+|y|)^{-l} \underset{\cap}{\overset{\cup}{}}$$
$$\underset{\cap}{\overset{\cup}{}} \|u\|\,(|y_m| + |y_{m+1}|)^{\gamma-\alpha}(1+|y'| + |y_m| + |y_{m+1}|)^{-l};$$

$$b)\ \ \forall x\in R_m^+,\quad \forall y\in\omega_x'$$
$$\left|u\left(y',\sqrt{y_m^2+y_{m+1}^2}\right)-u\left(x\right)\right|<c\left\|u\right\|\rho^{-1}\left(x\right)\ d^{\gamma}\left(\tilde{x},y\right)\ \substack{\cup\\ \cap}$$
$$\substack{\cup\\ \cap}\left\|u\right\|\rho^{-1}\left(x\right)\left(\left|\tilde{x}-y\right|\ /\ \left(1+\left|x\right|\right)^2\right)^{\gamma}.$$

3. Existence of a singular integral.

Let $f\left(\theta\right),\ \theta\in S_m^+$ be bounded and $u\in H_{\alpha\beta}^{\gamma}$ and (5) fulfilled.

Let us take $x\in R_m^+$ and fix it. We prove the absolute convergence of integrals

$$i_1\left(x;\omega_x'\right)=\int\limits_{\omega_x'}\frac{f(\tilde{\theta})}{r_{\tilde{x}y}^{m+2v}}\left(u\left(y',\sqrt{y_m^2+y_{m+1}^2}\right)-u\left(x\right)\right)\left|y_{m+1}\right|^{2v-1}dy\ .$$
$$i_2\left(x\right)=\int\limits_{R_{m+1}\backslash\omega_x'}\frac{f(\tilde{\theta})}{r_{\tilde{x}y}^{m+2v}}u\left(y',\sqrt{y_m^2+y_{m+1}^2}\right)\left|y_{m+1}\right|^{2v-1}dy\ .$$

Taking into account b) of corollary 1 we obtain

$$\left|i_1\left(x;\omega_x'\right)\right|\le c\left\|u\right\|\rho^{-1}\left(x\right)\left(1+\left|x\right|\right)^{-2\gamma}\left\|f\right\|J\left(x\right),$$

where $\left\|f\right\|=\sup\left|f\left(\theta\right)\right|,\ \theta\in S_m^+$,

$$J\left(x\right)=\int\limits_{\omega_x'}\left|y_{m+1}\right|^{2v-1}r_{\tilde{x}y}^{m+2v-\gamma}dy\le$$
$$\le c\int\limits_{\omega_x'}dy_1....dy_m\left(\int\limits_{\left\{y_{m+1}:\left|y_{m+1}\right|<\frac{x_m}{2}\right\}}\frac{\left|y_{m+1}\right|^{2v-1}dy_m}{\left(\sum\limits_{i=1}^{m}\left|y_i-x_i\right|^2+y_{m+1}^2\right)^{\frac{m+2v-\gamma}{2}}}\right)<x_m^{\gamma}.$$

$$\left|i_1\left(x;\omega_x'\right)\right|<\left\|f\right\|\ \ \left\|u\right\|\Psi_{\gamma}\left(x\right).\tag{6}$$

Whence

Subject to corollary 1 we obtain

$$L\left(\tilde{x},y\right)=\frac{\left|y_{m+1}\right|^{2v-1}}{\left|\tilde{x}-y\right|^{m+2v}}\left(\left|y_m\right|+\left|y_{m+1}\right|\right)^{\gamma-\alpha}\left(1+\left|y\right|\right)^{-l}\ \substack{\cup\\ \cap}$$
$$\substack{\cup\\ \cap}\frac{\left|y_{m+1}\right|^{2v-1}\left(\left|y_m\right|+\left|y_{m+1}\right|\right)^{\gamma-\alpha}}{\left(\left|x'-y'\right|+x_m+\left|y_m\right|+\left|y_{m+1}\right|\right)}\times\tag{7}$$
$$\times\left(1+\left|y'\right|+\left|y_m\right|+\left|y_{m+1}\right|\right)^{-l},\quad y\in R_{m+1}\backslash\omega_x'\ .$$

Let us introduce spaces

$$A_x=\left\{y\in\mathbf{R}_{m+1}:\left|\tilde{x}-y\right|\le\frac{\left|y\right|}{2}\right\},$$
$$B_x=\left\{y\in\mathbf{R}_{m+1}:\frac{\left|y\right|}{2}<\left|\tilde{x}-y\right|\le3\left|y\right|\right\},$$
$$C_x=\left\{y\in\mathbf{R}_{m+1}:3\left|y\right|<\left|\tilde{x}-y\right|\right\}\ .$$

Denote

$$i_2(x;G)=\int\limits_{(R_{m+1}\backslash\omega'_x)\cap G} L(\tilde{x},y)\,dy\ ,\qquad G\subset \mathbf{R}_{m+1}$$

Repeating the above mentioned reasonings, we obtain

$$\begin{gathered}|i_2(x)|<\|u\|\ \ \|f\|\,i_2(x;R_{m+1}\backslash\omega'_x)=\\ =\|u\|\ \ \|f\|\,(i_2(x;A_x)+i_2(x;B_x)+i_2(x;C_x))\ .\end{gathered}\tag{8}$$

Let us majorize the integrals in the right-hand side of (8).

Recalling that for $y\in A_x$ $(1+|x|)\underset{\cap}{\overset{\cup}{}}(1+|y|)$, subject to (7) we obtain

$$i_2(x;A_x)\le c\,(1+|x|)^{-l}\int\limits_0^\infty y_{m+1}^{2v-1}dy_{m+1}\int\limits_0^\infty y_{m+1}^{2v-1}dy_{m+1}\int\limits_0^\infty (y_m+y_{m+1})^{\gamma-\alpha}\,dy_m\times$$

$$\times\int\limits_{R_{m-1}}(|z|+x_m+y_m+y_{m+1})^{-m-2v}\,dz\ .$$

Hence,

$$i_2(x;A_x)<(1+|x|)^{-l}\,x_m^{\gamma-\alpha}=\Psi_\gamma(x)\,.\tag{9}$$

Let us majorize $i_2(x;B_x)$.

Let $|x|\ge 1$. Then for $y\in B_x$ $|\tilde{x}-y|\underset{\cap}{\overset{\cup}{}}|y|$ and $|y|\ge|x|/4$, finally $|\tilde{x}-y|\underset{\cap}{\overset{\cup}{}}|y|+|x|\underset{\cap}{\overset{\cup}{}}|y|+1$.

Taking this into account we have:

$$i_2(x;B_x)\le\int\limits_{B_x}\frac{(|y_m|+|y_{m+1}|)^{\gamma-\alpha}\,|y_{m+1}|^{2v-1}}{(|y|+|x|)^{m+2v+l}}dy\ .$$

Suppose $\mu=(\gamma+\beta)+(1+\gamma-\alpha)$. By virtue of (5) $\mu>0$. Taking into account that $m+2v+l=(m-1)+\mu+2v$, we obtain from the latter:

$$\begin{gathered}i_2(x;B_x)<\int\limits_0^{+\infty}y_{m+1}^{2v-1}dy_{m+1}\int\limits_0^{+\infty}(y_m+y_{m+1})^{\gamma-\alpha}\,dy_m\times\\ \times\int\limits_{R_{m-1}}(|z|+x_m+y_m+y_{m+1})^{m-1+\mu+2v}\,dz<\Psi_\gamma(x)\qquad(|x|\ge 1)\ .\end{gathered}$$

Let $|x|<1$ and $y\in B_x$. Then for $|y|\ge 1$ $|y|\underset{\cap}{\overset{\cup}{}}|y|+1\underset{\cap}{\overset{\cup}{}}|y|+1+|x|$ and also for $|y|<1$ $(1+|y|)\underset{\cap}{\overset{\cup}{}}1$ and $|x-y|\underset{\cap}{\overset{\cup}{}}|y|+x_m\underset{\cap}{\overset{\cup}{}}|y'|+|y_m|+|y_{m+1}|+x_m$.

Therefore

$$i_2(x;B_x)\le c\int\limits_{\{y\in R_{m+1}:|y|<1\}}\frac{(|y_m|+|y_{m+1}|)^{\gamma-\alpha}\,|y_{m+1}|^{2v-1}}{(|y'|+|y_m|+|y_{m+1}|+x_m)^{m+2v+l}}dy+$$

$$+\int\limits_{\{y\in\mathbf{R}_{m+1}:|y|\ge 1\}}\frac{(|y_m|+|y_{m+1}|)^{\gamma-\alpha}\,|y_{m+1}|^{2v-1}}{(|y|+1+|x|)^{m+2v+l}}dy\prec$$

$$\prec\left(x_m^{\gamma-\alpha}+\tfrac{1}{(1+|x|)^{\beta+\gamma}}\right)<\Psi_\gamma(x)\qquad(|x|\prec 1)\ .$$

Thus, we proved that

$$i_2(x;B_x) \prec \Psi_\gamma(x) \tag{10}$$

The validity of estimate (10) for $i_2(x;C_x)$ is proved by analogous reasonings.

Thus, we proved that

$$\begin{gathered} i_2(x;R_m\backslash\omega'_x) < \Psi_\gamma(x) \text{ and} \\ |i_2(x)| < \|u\| \ \ \|f\| \, \Psi_\gamma(x) \end{gathered} \tag{11}$$

So, the absolute convergence of integrals $i_1(x)$, $i_2(x)$ is proved.

Theorem A. *Let $u \in H^\gamma_{\alpha\beta}$ and (5) be fulfilled. If $f(\theta)$, $\theta \in S^+_m$ is bounded and*

$$\int\limits_{S^+_m} f(\theta)\,\theta^{2v}_m ds(\theta) = 0\ . \qquad (*)$$

then at each point $x \in R^+_m$ there exists SI $Au(x)$ generated by GSI T^y, and the following equality holds

$$Au(x) = v.p. \int\limits_{\mathbf{R}^+_m} f(\theta)\,|S|^{-m-2v}\left[T^Y u(x)\right] S^{2v}_m ds =$$

$$= \tfrac{1}{2}C_v \int\limits_{\omega'_x} f\left(\tilde\theta\right)|\tilde x - y|^{-m-2v}\left(u\left(y';\sqrt{y^2_{m+1}+y^2_m}\right) - u(x)\right)|y_{m+1}|^{2v-1}dy + \tag{3$\prime$}$$

$$+\tfrac{1}{2}C_v \int\limits_{R_{m+1}\backslash\omega'_x} f\left(\tilde\theta\right)|\tilde x - y|^{-m-2v} u\left(y';\sqrt{y^2_{m+1}+y^2_m}\right)|y_{m+1}|^{2v-1}dy\ .$$

Proof. From (4) by virtue of (*) we obtain

$$\int\limits_{S_{m+1}} f\left(\tilde\theta\right)|\theta_{m+1}|^{2v-1} \qquad dS(\theta) = 0\ . \tag{**}$$

Let $u \in H^\gamma_{\alpha\beta}$, $x = (x', x_m) \in R^+_m$, $0 < \varepsilon < \frac{x_m}{2}$. Then from (3) we obtain

$$A_\varepsilon u(x) = \tfrac{1}{2}C_v\left(\int\limits_{S_{m+1}} f\left(\tilde\theta\right)|\theta_{m+1}|^{2v-1}dS(\theta)\right)[u(x)]\ln\frac{\chi_m}{2\varepsilon} +$$

$$+\tfrac{1}{1}C_v i_1(x;\omega'_x(\varepsilon)) + i_2(x)\ ,$$

where $\omega'_x(\varepsilon) = \left\{y \in R_{m+1} : \varepsilon < |\tilde x - y| < \dfrac{x_m}{2}\right\}$.

Now taking into account (**) and the absolute convergence of integrals $i_1(x;\omega'_x)$ and $i_2(x)$, passing to the limit as $\varepsilon \to +0$, we prove the theorem

4. Boundedness in $H^\gamma_{\alpha\beta}$.

Theorem C. *Let f satisfy condition (*) and*

$$|f(\theta_1) - f(\theta_2)| \le c_f\,|\theta_1 - \theta_2|^\delta$$

where C_f is a constant, $\theta_1, \theta_2 \in S_m^+$ and $0 < \delta \leq 1$. If $0 < \gamma < \delta \leq 1$, $0 < \alpha - \gamma < 1$, $\beta + \gamma < m$, then SI operator generated by GSO T^Y:

$$A : u \to Au(x) \equiv v.p. \int\limits_{R_m^+} f(\theta) |s|^{-m-2v} \left[T^S u(x)\right] S_m^{2v} ds$$

is bounded in $H_{\alpha\beta}^{\gamma}$.

Proof. By virtue of theorem A from (6) and (11) we obtain

$$|Au(x)| \leq \frac{1}{2} C_v C \left(\left|i_1 \left(x; \omega'_x\right)\right| + |i_2(x)|\right) < \|f\| \ \ \|u\| \, \Psi_\gamma(x). \tag{12}$$

By virtue of lemma 2, to prove the theorem it suffices to show that $\forall x \in R_m^+$ and $|h| \leq x_m/8$

$$|Au(x) - Au(x+h)| \leq c \|u\| \rho^{-1}(x) (1 + |x|)^{-2v} |h|^v \qquad (13) \tag{13}$$

where c is undependent of x and h.

Suppose

$$\omega_1(x) = \omega(\tilde{x}, 2|h|), \ \omega_2(x) = \omega\left(\widetilde{x+h}, 3|h|\right),$$
$$\omega_3(x) = \omega\left(\widetilde{x+h}, \frac{x_m}{2} - |h|\right).$$

Obviously $\omega_1(x) \subset \omega_2(x) \subset \omega_3(x) \subset \omega'_x$.

Subject to (*) and (4) one can prove that

$$Au(x) - Au\left(\widetilde{x+h}\right) = \sum_{i=1}^{5} J_i(x; h), \tag{14}$$

where

$$J_1(x; h) = \left(\int\limits_{\omega_2} + \int\limits_{\omega'_x \backslash \omega_3}\right) K(y, \tilde{x}) (u_1(y) - u(x)) |y_{m+1}|^{2v-1} dy,$$

$$J_2(x; h) = -\int\limits_{\omega_2} K\left(y, \widetilde{x+h}\right) (u_1(y) - u(x)) |y_{m+1}|^{2v-1} dy,$$

$$J_3(x; h) = -\int\limits_{\omega'_x \backslash \omega_3} K\left(y, \widetilde{x+h}\right) u_1(y) |y_{m+1}|^{2v-1} dy,$$

$$J_4(x; h) = \int\limits_{\omega'_x \backslash \omega_3} \left(K(y, \tilde{x}) - K\left(y, \widetilde{x+h}\right)\right) (u_1(y) - u(x)) |y_{m+1}|^{2v-1} dy,$$

$$J_5(x; h) = \int\limits_{R_{m+1} \backslash \omega'_x} \left(K(y, \tilde{x}) - K\left(y, \widetilde{x+h}\right)\right) u_1(y) |y_{m+1}|^{2v-1} dy,$$

where $K(y,\tilde{x}) = f\left(\tilde{\theta}\right) \big/ r_{y\tilde{x}}^{m+2v}$, $r_{y\tilde{x}} = |\tilde{x} - y|$,

$$u_1(y) = u\left(y', \sqrt{y_m^2 + y_{m+1}^2}\right).$$

Using easy calculations, one can prove that for $x \in \mathbf{R}_m^+$, $y \in R_{m+1} \backslash \omega_2$ and $|h| \leq x_m \backslash 8$

$$r_{y\tilde{x}} \underset{\cap}{\overset{\cup}{}} r_{y\,\widetilde{x+h}}; \tag{15}$$

$$\left|K(y,\tilde{x}) - K\left(y, \widetilde{x+h}\right)\right| \prec \left(c_f |h|^{\delta} r_{y\tilde{x}}^{-(m+2v+\delta)} + \right.$$
$$\left. + \|f\| \ |h| \, r_{y\tilde{x}}^{-(m+2v+1)}\right) \prec (c_f + \|f\|) |h|^{\delta} r_{y\tilde{x}}^{-(m+2v+\delta)} . \tag{16}$$

Not let us majorize $|J_i(x;h)|$ $\quad i = \overline{1,5}$.

Taking into account b) of corollary 1 and also (15), we obtain

$$|J_1(x;h)| \leq c \left(\int\limits_{\omega_2} + \int\limits_{\omega'_x \backslash \omega_3} \right) \frac{|f\left(\tilde{\theta}\right)|}{|y - \tilde{x}|^{m+2v}} \rho^{-1}(x) \left(\frac{|y - \tilde{x}|}{(1+|x|)^2} \right)^{\gamma} |y_{m+1}|^{2v-1} dy \prec$$

$$\prec c_f \rho^{-1}(x) (1+|x|)^{-2\gamma} \left(\int\limits_{\omega_2} + \int\limits_{\omega'_x \backslash \omega_3} \right) \frac{|y_{m+1}|^{2v-1}}{|\widetilde{(x+h)} - y|^{m+2v-\gamma}} \prec$$

$$\prec c_f \rho^{-1}(x) (1+|x|)^{-2\gamma} |h|^{\gamma} .$$

The following expression is proved analogously

$$|J_2(x;h)| < c_f p^{-1}(x) (1+|x|)^{-2\gamma} |h|^{\gamma};$$

$$J_3(x;h) \leq c c_f \|u\| \int\limits_{\omega'_x \backslash \omega_3} \frac{(|y_m| + |y_{m+1}|)^{\gamma - \alpha}}{r_{y\widetilde{x+h}}^{m+2v} (1+|y|)^{l}} |y_{m+1}|^{2v-1} dy \prec$$

$$\prec \|f\| \ \|u\| \, \Psi_{\gamma}(x) \int\limits_{\omega'_x \backslash \omega_3} \frac{|y_{m+1}|^{2v-1}}{r_{y\widetilde{x+h}}^{m+2v}} dy .$$

Taking into account that $A = \left\{ y \in R_{m+1} : |\widetilde{x+h} - y| < \dfrac{x_m}{2} + \dfrac{|h|}{2} \right\} \supset \omega'_x$ and passing to the polar coordinates, we obtain

$$\int\limits_{\omega'_x \backslash \omega_3} \frac{|y_{m+1}|^{2v-1}}{r_{y\widetilde{x+h}}^{m+2v}} dy \ \leq \int\limits_{A \backslash \omega_3} \frac{|y_{m+1}|^{2v-1}}{r_{y\widetilde{x+h}}^{m+2v}} dy \ \leq c \frac{|h|}{x_m} .$$

Hence

$$|J_3(x;h)| < \|f\| \ \ \|u\| \, \Psi_\gamma(x) \, |h| x_m^{-1} < J_4(x;h) \, \rho^{-1}(x) \, (1+|x|)^{-2v} \, |h|^v \ .$$

Let us majorize $J_4(x;h)$. Taking into account (13) and (15), we obtain

$$|J_4(x;h)| \le c \, (c_f + \|f\|) \, \|u\| \int\limits_{\omega_3 \backslash \omega_2} \frac{|h|^\delta}{r_{y\widetilde{x+h}}^{m+2v+\delta}} \rho^{-1}(x) \left(\frac{r_{y\widetilde{x+h}}}{(1+|x|)^2} \right)^\gamma |y_{m+1}|^{2v-1} dy \prec$$

$$\prec (c_f + \|f\|) \, \|u\| \left(\frac{|h|^\delta}{(1+|x|)^{2\gamma}} \right) \rho^{-1}(x) \, |h|^{\gamma-\delta} \ .$$

Let us majorize $|J_5(x;h)|$. Subject to (16) and (15) we obtain

$$|J_5(x;h)| \le c \, (c_f + \|f\|) \, \|u\| \int\limits_{R_{m+1} \backslash \omega'_x} \frac{|h|^\delta}{r_{y_{y\tilde{x}}}^{m+2v+\delta}}$$

$$\frac{(|y_m| + |y_{m+1}|)^{\gamma-\alpha}}{(1+|y'|+|y_m|+|y_{m+1}|)^l} |y_{m+1}|^{2v-1} dy \prec$$

$$\prec (c_f + \|f\|) \, |h|^\delta x_m^{-\delta} i_2 (x; R_{m+1} \backslash \omega'_x) \prec$$

$$\prec (c_f + \|f\|) \, (|h|/x_m)^\delta \, \Psi_\gamma(x) < (c_f + \|f\|) \, (|h|/x_m)^\gamma \, \Psi_\gamma(x) \prec$$

$$\prec (c_f + \|f\|) \, \|u\| \, \rho^{-1}(x) \, (1+|x|)^{-2\gamma} \, |h|^\gamma \ .$$

Thus, theorem C is completely proved.

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