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DETERMINATION OF EXTREMALS ON THE BOUNDARY OF DOMAIN

Abstract

The formula for computation of the best approximation is established and the best approximate function is constructed in approximation of two-variable function on the boundary of unit square by the functions of the form $\varphi(x)+\psi(y)$.

In the paper [1] as far as we know, at first we succeeded in finding the extremals, namely, in constructing the best approximate function and computing the value of the best approximation in the domain consisting of unit square. Some parts of positive plane measure are not inside of this square. It was a question of approximation of two-variable function by the sums of one-variable function. At that it is assumed the existence of solution of a functional equation associated with approximated function. As should be expected the extremal function in this approximation depended on the solution of this equation.

In this paper the formula for computation of the best approximation is established and the best approximate function is constructed in approximation of two-variable function on the boundary of unit square by the functions of the form $\varphi(x)+\psi(y)$.

Consider an arbitrary set $Q\subset K$, where $K=I^2,\ I=[0,1]$ is a unit square. Denote by $M(Q)$ a set of the functions $f=f(x,y)$ for the arbitrary points (x',y'), (x'',y'), (x',y''), (x'',y''), $x'<x''$, $y'<y''$ from Q satisfying the condition

$$f\left(x'',y''\right)+f\left(x',y'\right)-f\left(x'',y'\right)-f\left(x',y''\right)\geq 0. \qquad (*)$$

Since, these points are vertices of the rectangle $\Pi=[x',x'';y',y'']\subset K$, then it geometrically means that nonnegative quantity is associated to every rectangle $\Pi\subset K$ with the vertices from Q, although the rectangle Π itself need not belong to the set Q.

Let the function $f=f(x)$ be determined on the boundary of the unit square K, namely, on the set

$$\Pi^0=\{0\leq x\leq 1;\ y=0,1\}\cup\{x=0,1;\ 0\leq y\leq 1\}.$$

The functions $\varphi(x)$ and $\psi(y)$ are determined at $0\leq x\leq 1$ and $0\leq y\leq 1$, respectively.

Consider the best approximation of the function f on the boundary of the square K

$$E_f=E\left[f,\varphi+\psi,\Pi^0\right]=\inf_{\varphi+\psi}\sup_{(x,y)\in\Pi^0}|f-\varphi-\psi|=\inf_{\varphi+\psi}\|f-\varphi-\psi\|_{C(\Pi^0)}. \qquad (1)$$

We call the function $\varphi_0(x)+\psi_0(y)$ satisfying the equality

$$\|f-\varphi_0-\psi_0\|=E_f$$

the best approximating in approximation (1).

Theorem. *For the function* $f \in M\left(\Pi^0\right)$ *the best approximation* E_f *may be calculated by the formula*

$$E_f = \frac{1}{4}\left[f(1,1) + f(0,0) - f(1,0) - f(0,1)\right],$$

and the sum

$$\varphi_0(x) + \psi_0(y) = \tfrac{1}{2}\left[f(x,0) + f(x,1) + f(0,y) + f(1,y)\right] -$$

$$-\tfrac{1}{4}\left[f(1,1) + f(0,0) + f(0,1) + f(1,0)\right]$$

is the best approximating in approximation (1).

Proof. For arbitrary rectangle $\Pi' = [x', x'', y', y'']$ with the vertices from Π^0 we denote

$$L\left(f, \Pi'\right) = L_f\left(\Pi'\right) = \frac{1}{4}\left[f\left(x'', y''\right) + f\left(x', y'\right) - f\left(x'', y'\right) - f\left(x', y''\right)\right]. \tag{2}$$

We have $L_{\varphi+\psi}(\Pi') = 0$.

Then by virtue of the linearity of the functional L

$$\begin{gathered} L(f, \Pi') = L(f, \Pi') - L\left[\varphi(x) + \psi(y), \Pi'\right] = \\ = L\left[f(x,y) - \varphi(x) - \psi(y),\ Q\right] \le \left\|f(x,y) - \varphi(x) - \psi(y)\right\|_{C(\Pi')} \end{gathered} \tag{3}$$

and as the left hand side of inequality (3) does not depend on $\varphi(x) + \psi(y)$, then

$$L\left(f, \Pi'\right) \le \inf_{\varphi+\psi} \left\|f(x,y) - \varphi(x) - \psi(y)\right\|_{C(\Pi')}.$$

Π' is arbitrary rectangle with the vertices from Π^0, therefore, from the last relation we obtain

$$L\left(f, \Pi^0\right) \le \inf_{\varphi+\psi} \|f - \varphi - \psi\|_{C(\Pi^0)} = E_f. \tag{4}$$

Relation (4) is a lower estimation of the best approximation (1).

For the rectangle $\Pi = [x_1, x_2; y_1, y_2]$ with the vertices from Π^0, we introduce the denotation

$$L_f(\Pi) = \begin{bmatrix} x_2 & y_2 \\ x_1 & y_1 \end{bmatrix}_f \tag{5}$$

and we consider the function

$$F(x,y) = \begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}_f - \begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix}_f - \begin{bmatrix} 1 & y \\ x & 0 \end{bmatrix}_f + \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}_f \tag{6}$$

determined on the set Π^0.

By virtue of denotation (5) and determination (2) we have

$$\begin{gathered} F(x,y) = f(x,y) - \frac{1}{2}\left[f(x,1) + f(x,0) + f(1,0) + f(0,y)\right] + \\ +\frac{1}{4}\left[f(1,1) + f(0,0) + f(1,0) + f(0,1)\right] \stackrel{def}{=} f(x,y) - \varphi_0(x) - \psi_0(y). \end{gathered} \tag{7}$$

Further, it is clear that

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f = \frac{1}{4}\left[f(1,1) + f(0,0) - f(1,0) - f(0,1)\right] =$$

$$= \begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}_f - \begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix}_f - \begin{bmatrix} 1 & y \\ x & 0 \end{bmatrix}_f + \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}_f . \tag{8}$$

For the simplification of notation later on we introduce the denotation

$$\Sigma_{e;f} = \begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}_f + \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}_f ; \quad \Sigma_{o;f} = \begin{bmatrix} 1 & y \\ x & 0 \end{bmatrix}_f + \begin{bmatrix} 1 & y \\ x & 0 \end{bmatrix}_f . \tag{9}$$

Using notation (9) according to determination of the function $F(x,y)$

$$F(x,y) = \Sigma_{e;f} - \Sigma_{o;f} . \tag{10}$$

Then from (8)-(10) we obtain

$$\Sigma_{e;f} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \Sigma_{o;f},$$

$$\Sigma_{o;f} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \Sigma_{e;f} .$$

Substituting these values in (10) we have

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f - 2\,\Sigma_{o;f}$$

$$F = 2\Sigma_{e;f} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f . \tag{11}$$

According to the definition of the class $M\left(\Pi^0\right)$

$$\Sigma_{e;f} \geq 0, \ \Sigma_{o;f} \geq 0.$$

Then from (11) we obtain

$$\left.\begin{array}{l} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f - F = 2\,\Sigma_{o;f} \ \geq 0 \\ \\ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f + F = 2\,\Sigma_{e;f} \ \geq 0 \end{array}\right\} \Longrightarrow -\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f \leq F \leq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f . \tag{12}$$

Now, using determination of the function F from (6) we calculate its values at the points $(1,1)$ and $(1,0)$

$$F(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_f - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_f - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_f + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f ,$$

$$F(1,0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_f - \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}_f - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_f = -\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_f .$$

These last calculations show that the values of the function $F(x,y)$ achieve boundaries in inequalities (12), whence we have

$$\|F\|_{C(\Pi_0)}=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}_f .$$

But according to (7) the function $F(x,y)$ has form $f(x,y)-\varphi_0(x)-\psi_0(y)$.
Thus,

$$\|F\|=\|f(x,y)-\varphi_0(x)-\psi_0(y)\|_{C(\Pi_0)}=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}_f=L_f\left(\Pi^0\right)$$

and by virtue of lower estimation (4)

$$\|f(x,y)-\varphi_0(x)-\psi_0(y)\|\le E_f .$$

According to the definition of the best approximation there cannot be sign of inequality in the last relation. Consequently,

$$E_f=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}_f=\frac{1}{4}\left[f(1,1)+f(0,0)-f(1,0)-f(0,1)\right]$$

and the function

$$\varphi_0(x)+\psi_0(y)=f(x,y)-F(x,y)=$$

$$=\frac{1}{2}\left[f(x,0)+f(x,1)+f(0,y)+f(1,y)\right]- \tag{13}$$

$$-\tfrac{1}{4}\left[f(1,1)+f(0,0)+f(1,0)+f(0,1)\right]$$

is the best approximating in the considered approximation. The theorem is proved.

2. Let the function f be determined on the whole square K. Then, it is clear that for its approximation by sums of the functions of the form $\varphi(x)+\psi(y)$, theorem 1 is valid on the boundary of the square K. Let now Q be an arbitrary subset of the square K, containing the boundary of square

$$\Pi^0\subset Q\subset K.$$

It is clear that

$$\|f-\varphi-\psi\|_{C(\Pi^0)}\le\|f-\varphi-\psi\|_{C(Q)}\le\|f-\varphi-\psi\|_{C(K)} . \tag{14}$$

By virtue of the fact that the function $\varphi(x)$ is determined for all $0\le x\le 1$, and the function $\psi(y)$ is determined for all $0\le y\le 1$, then the sum $\varphi(x)+\psi(y)$ is determined on the whole K and consequently, on its any subset Q. Therefore, from (14) we obtain

$$\inf_{\varphi+\psi}\|f-\varphi-\psi\|_{C(\Pi^0)}\le\inf_{\varphi+\psi}\|f-\varphi-\psi\|_{C(Q)}\le\inf_{\varphi+\psi}\|f-\varphi-\psi\|_{C(K)} . \tag{15}$$

Denote by $E[f,Q]$ and $E[f,K]$ the best approximations of the function f by the functions of the form $\varphi(x)+\psi(y)$ on the sets Q and K, respectively. Then, relation (15) may be written in the following form

$$E\left[f,\Pi^0\right] \le E[f,Q] \le E[f,K].$$

In [2] we established that the best approximation of the functions $f \in M(K)$ by sums of the form $\varphi(x)+\psi(y)$ may be calculated by the formula

$$E[f,K] = \frac{1}{4}\left[f(1,1)+f(0,0)-f(1,0)-f(0,1)\right],$$

and the function $\varphi_0(x)+\psi_0(y)$ of the form (13) is the best approximating in this approximation. Comparing this result with theorem 1 we obtain a result for the arbitrary set Q, satisfying the inequality $\Pi^0 \subset Q \subset K$, namely, the following theorem is valid.

Theorem 2. *Let Q be a subset of the square K and contain its boundary. Then for the arbitrary function $f \in M(Q)$, the basic extremals of the bets approximation $E[f,Q]$ are determined by the following form: the best approximation is calculated by the formula*

$$E[f,Q] = \frac{1}{4}\left[f(1,1)+f(0,0)-f(1,0)-f(0,1)\right]$$

and the sum

$$\varphi_0(x)+\psi_0(y) = \frac{1}{2}\left[f(x,0)+f(x,1)+f(0,y)+f(1,y)\right]-$$

$$-\frac{1}{4}\left[f(1,1)+f(0,0)+f(1,0)+f(0,1)\right]$$

is the best approximating in this approximation.

References

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