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ON GREEN FUNCTION AND DISTRIBUTION OF EIGENVALUES OF THE SECOND ORDER PARTIAL OPERATOR- DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE IN HALF-SPACE

Abstract

Operator L generated by the expression

$$l(u) = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + Q(x)u$$

and the boundary condition

$$u(x_1, x_2, x_3)|_{x_3=0} = 0$$

is considered in the Hilbert space $L_2\left(E_3^+,\ H\right)$.

Under some assumptions relative to the coefficients $a_{ij}(x)$ and operator potential $Q(x)$ Green function is constructed, the discreteness of the spectrum is proved and the asymptotic formula for distribution function of eigenvalues of operator L is obtained.

Let E_3^+ be half-space $(x_3 \geq 0)$ of three- dimensional Euclidean space E_3, H be a separable Hilbert space. Consider the following differential expression in the Hilbert space $L_2\left(E_3^+,\ H\right)$

$$l(u) = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + Q(x)u \tag{1}$$

with the boundary condition

$$u(x)|_{x_3=0} = u(x_1, x_2, x_3)|_{x_3=0} = 0 \tag{2}$$

It is assumed the fulfillment of the following conditions relative to the coefficients $a_{ij}(x)$ and operator function $Q(x)$:

1) Real-valued functions $a_{ij}(x) = a_{ji}(x)$ have bounded partial derivatives $\dfrac{\partial a_{ij}(x)}{\partial x_k}$ $(i, j, k = 1, 2, 3)$ on E_3^+, moreover, the conditions of uniform ellipticity are satisfied, i.e. there exist $m, M > 0$ such that

$$m|\xi|^2 \leq \sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \leq M|\xi|^2 .$$

2) For each $x \in E_3^+$ operators $Q(x)$ have common everywhere dense domains and they are self-adjoint operators, $Q(x) \geq E$ and $Q^{-1}(x) \in \sigma_\infty$.

3) For $|x - \xi| \leq 1$,

$$\left\|\left[Q(x) - Q(\xi)Q^{-a}(x)\right]\right\| \leq B|x - \xi| .$$

4) For some $l>0$ $Q(x)\in\sigma_1$ for all $x\in E_3^+$ and

$$\int\limits_{E_3^+}\left\|Q^{-l}(x)\right\|_1 dx<\infty.$$

5) There exists such a function f that for $|x-\xi|\le 1$,

$$\left\|e^{-ctQ(\xi)}\right\|_1\le\left\|e^{-f(c)tQ(\xi)}\right\|_1$$

for all $c>0,\ t>0.$

6) For any fixed $c>0$,

$$\int\limits_{E_3^+} tre^{-ctQ(x)}dx=O(1)\int\limits_{E_3^+} tre^{-tQ(x)}dx.$$

7) Let $\alpha_1(x)\le\alpha_2(x)\le...\le\alpha_n(x)\le...$ be eigenvalues of the operator $Q(x)$ in H. Suppose that $\alpha_1(x),\alpha_2(x),...,\alpha_n(x),...$ are measurable functions.

We will introduce the denotation

$$\rho(\lambda)=\sum_{i=1}^{\infty}\int\limits_{\{x:\alpha_i(x)<\lambda\}}\Phi(x)\left[\lambda-\alpha_i(x)\right]^{\frac{3}{2}}dx,$$

where $\Phi(x)=\int\limits_{E_3^+} e^{-\sum\limits_{i,j=1}^{3}a_{ij}(x)\xi_i\xi_j}d\xi=\pi^{\frac{3}{2}}\left(\det\left\|a_{ij}(x)\right\|_{i,j=1}^{3}\right)^{-\frac{1}{2}}.$

Suppose that for some $a_0>0$ and sufficiently large $\lambda>0$ the tauberian condition is fulfilled

$$\lambda\rho'(x)<a_0\rho(\lambda)\qquad(3)$$

If conditions 1)-7) are satisfied then it is proved that operator L generated by the expression (1) and boundary condition (2) has a discrete spectrum. Let $\lambda_1,\lambda_2,...,\lambda_n,...$ be eigenvalues of the operator L.

In the given paper the distribution of eigenvalues, i.e. asymptotic behaviour of distribution function $N(\lambda)$ as $\lambda\to\infty$ is studied. By definition, function $N(\lambda)=\sum\limits_{\lambda_n>\lambda}1$ shows the number of eigenvalues less than λ.

With this purpose, at first, Green function of parabolic problem with the operator L is studied:

$$\frac{\partial u}{\partial t}=-Lu=-\left[L_0\left(x,\frac{\partial}{\partial x}\right)+L_1\left(x,\frac{\partial}{\partial x}\right)+Q(x)\right]u,$$

$$u(0,x)=\psi(x),\ x\in E_3^+,\ \psi(x)\in L_2\left(E_3^+;H\right).$$

One of the main results is

Theorem 1. *If the coefficients of the differential expression (1) satisfy conditions 1)-5), then for $t\to+0$ the following asymptotic formula holds*

$$G(x,y,t)=G_1(x-y,y,t)\qquad(4)$$

where $O(1)$ *is an operator from* σ_1 *for each* $x, y \in E_3^+$ *and small* $t > 0$*, whose* σ_1*-norm is bounded on* x, y, t*.*

Here

$$G_1(x-y, y, t) = R(x-y) G_0(x-y, y, t),$$

$R(x)$ *is some smooth function satisfying the condition*

$$R(x) = \begin{cases} 1 \; if & |x| \le \frac{1}{2}, \\ 0 \;\; if & |x| > 1. \end{cases}$$

$G(x-y, \eta, t)$ *is Green function of the problem with "frozen" coefficients*

$$\frac{\partial u}{\partial t} = -L\left(\eta, \frac{\partial}{\partial x}\right) u,$$

$$u(0, x) = \psi(x), \; \psi(x) \in L_2\left(E_3^+, \; H\right).$$

This theorem particularly implies that the spectrum of the operator L is purely discrete one.

By means of theorem 1 and tauberian Keldysh M.V. theorem [3] the following main theorem is proved.

Theorem 2. *If the coefficients* $a_{ij}(x)$ *and the operator function* $Q(x)$ *satisfy the conditions 1)-7) and tauberian condition (3) is fulfilled, then for the number of eigenvalues* $N(\lambda)$ *of the operator* L *as* $\lambda \to \infty$ *the following asymptotic formula holds*

$$N(\lambda) \tilde{} \frac{1}{(2\pi)^3 \Gamma\left(\frac{5}{2}\right)} \sum_{i=1}^{\infty} \int_{\{x : \alpha_i(x) < \lambda\}} \Phi(x) \left[\lambda - \alpha_i(x)\right]^{\frac{3}{2}} dx.$$

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