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# MODELING OF BELLMAN-HARRIS BRANCHING PROCESSES

**Abstract**

*In the present paper the Bellman-Harris branching random processes are investigated. Different properties of these processes are investigated and on the basis of obtained above theoretical results the noted processes are modeled in computer.*

Recently many papers appear which is devoted to stochastic analysis of branching random processes. In this aspect in [1] the mathematical model of Galton-Watson classical random process is constructed and this process is modeled in computer.

Further on the base of modeling the basic probability characteristics of indicated process are investigated.

In the present paper the analogous investigations are carried out for Bellman-Harris branching random processes. The mathematical construction of probability space on which the Bellman-Harris branching random processes are determined, is stated in detail in [2].

The Bellman-Harris processes differ from the Galton-Watson process only by the fact that the moment of appearance of particle up to its conversion or disappearance passes some random time, named as life time. Rather, assume, that the process begins from one particle of zero age having random time of life $l$ with distribution of the probability $G(t) = P\{l \leq t\}$. At the end of life the particle with the probabilities $p_k,\ k = 0, 1, 2, ...$ is converted to $k$ new particles of zero age. It is assumed, that the distribution of life time is the same for all particles and the probabilities $p_k$ doesn't depend on the age of particles at the moment of its conversion and from the number of other existent particles. The process continues until there exists even one particle.

Thus the Bellman-Harris branching processes can be determined by the distribution function of life time $G(t)$ and the generating function

$$h(z) = \sum_{k=0}^{x} p_k z^k$$

of the number of direct descendants of one particle.

The Bellman-Harris process whose life time is identically equal to unit will be the Galton-Watson process. Therefore any Bellman-Harris branching random process is closely concerned with the corresponding Galton-Watson process, whose generating function of number of direct descendant of one particle is $h(z)$.

Let $\xi(t),\ t \geq 0$ be Bellman-Harris branching process with generating function of number of particles at the moment of time $t$

$$F(t; z) = \sum_{k=0}^{\infty} P\{\xi(t) = k\} z^k = M\left\{z^{\xi(t)} | \xi(0) = 1\right\}$$

It is evident [2], that the generating function $F(t;z)$ satisfies the following nonlinear integral equation

$$F(t;z) = z[1-G(t)] + \int_0^t h(F(t-u;z))\,dG(u). \tag{1}$$

If we search all possible functions for distribution life time of one particle $G(t)$ and all possible generating functions $h(z)$ (the generating functions of corresponding Galton-Watson processes), then among the obtained Bellman-Harris branching processes may be found such ones whose number of particles in every fixed moment of time $t>0$ with positive probability is equal to infinity. If there is no such phenomenon, then the process is called regular. The continuity in zero of distribution function of the life time $G(t)$, i.e., $G(0+)=0$ and the finiteness $m=\Sigma kp_k$ of average number of direct descendants of one particle are sufficient conditions of regularity.

As it is noted any Bellman-Harris branching process is closely connected with the corresponding Galton-Watson process. The degeneration of Bellman-Harris process is not connected generally with the life time of particle and consequently, the probability $q$ of degeneration of the Belman-Kharis process is equal to the nearest $o$ solution $z=h(z)$ in unit interval $0\le z\le l$.

The generating function $F(t;z)$ is connected with double inequalities with the generating function $h_n(z)$ of the number of particles in the $n$-th generation or the corresponding Galton-Watson process. Namely, if $0\le z\le q$, then

$$h_n(z) - (q-z)\sum_{k=0}^{n-1}\left[h_n'(q)\right]^k\left\{G^{k^*}(t) - G^{k+1^*}(t)\right\} \le$$

$$\le F(t;z) \le h_n(z) + (q-z)\left[h_n'(q)\right]^n G^{n^*}(t)$$

Here the natural number $n$ and nonnegative number $t$ are arbitrary, $G^{k^*}(t)$- $k$-multiple convolution of the distribution function $G$ with itself. In the critical case $(m=h(1)=1)$ this inequalities take the especially simple form. If $m=1$, then it is evident, that $q=1$ consequently,

$$h_n(z) - (1-z)\left[1-G^{n^*}(t)\right] \le F(t;z) \le h_n(z) + (1-z)G^{n^*}(t) \tag{2}$$

for all $0\le z\le 1,\ n=0,1,2,...\quad t\ge 0$.

In [3] using the double inequality (2) the limit theorem on convergence of normed Bellman-Harris processes to the Jirina critical branching process is proved (the branching process with continuous state space).

And in [4] applying the integral equation (1) the limit theorem on convergence of one-dimensional distributions of Bellman-Harrin branching processes to the Jirini one-dimensional distributions is proved.

In particular cases we can obtain integral equation of form (1) for the generating function $F(t;z)$ by means of which the probabilistic characteristics of the process are investigated.

Note, that by obtaining integral equation (1) it is assumed, that the any particle at the expiry of its life time with the definite probability $(p_k)$ is converted to the random number of new particles of zero age.

If we assume, that the particle on ending of life time generates two new particles, then for the generating function $F(t;z)$ the following integral equation is obtained

$$F(t;z) = z[1 - G(t)] + \int_0^t [F(t-u;z)]^2 \, dG(u). \qquad (3)$$

The evident generalization of this model is the assumption about that the particle at the expiry of its life time generates equally $r$ new particles of the same type, where $r$ is a fixed integer $r \geq 2$. In this case the integral equation (3) is substituted by the following one:

$$F(t;z) = z[1 - G(t)] + \int_0^t [F(t-u;z)]^r \, dG(u). \qquad (4)$$

In general case the integral equations (1), (3) and (4) aren't solved. Therefore for the investigation of the described processes we model their behaviour in computer and mark the compare with the results obtained by an analytical way.

Exactly as in [1] we assume, that the process begins from one particle of the zero age $\xi(0) = 1$ and the probability law of development of population has the following form

$$p_k = P\{\xi_1 = k\} = \frac{1}{2^k}, \; k = 1, 2, ... \qquad \sum_{k=1}^{\infty} p_k = 1$$

Besides, we assume, that the distribution function of the life time $G(t)$ has exponential form, i.e.,

$$G(t) = 1 - e^{-\lambda t}, \quad t \geq 0$$

By composing a program for the modelling of behaviour of Bellman-Harris branching processes in computer it is chosen pseudorandom number $\alpha$ from the interval $(0,1)$, and the parameter $\lambda$ is chosen equal to 2. If $\alpha$ is situated in the interval $\left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$, i.e., $p_{k+1} < \alpha \leq p_k$, then in the next generation the quantity of anew generated particles is equal to $k$. If $\frac{1}{2^k} < \alpha < 1$, then the number of particles in the next generation is equal to 0, i.e. the process is degenerated.

At this approach the following characteristics of Bellman-Harris branching processes was investigated:

-summary number of particles in the definite moment

- average number of particles in definite moment

- variance of amount of particles in definite moment.

It should be noted, that the indicated investigation is conducted for non-normalized processes. At this the 16 realizations of process are considered, and it is introduced only 6-th realization of the process in graphic for summary number of particles (fig.1).

Further, the analogous investigation is conducted for Bellman-Harris normalized processes.In this case the above mentioned characteristics of normalized processes are studied. The 16 realizations of the process are considered, but in graphics for summary number of particles (fig.4) it is introduced the 9-th realization of process. All results obtained at modelling in computer are shown in graphics and in tables. The summary numbers of particles in the system for non-normalized case (6-th realization) is shown in fig.1. The average and variance of amount of particles in the system for non-normalized case is shown in fig.2 and 3. The corresponding characteristics for normalized processes is shown in fig. 5 and 6.

At modelling it was chosen the time of modelling $T = 750$. Unlike the case for Galton-Watson branching processes, here during the time $T$ appears $n = 380$ generations. It is explained by the randomness of life time of particle (but the generation takes some time unit).

According to the graphics of process behaviour and in tables obtained at modelling in computer we can argue, that the results obtained by the analogous way in [3] and [4] are reliable

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**Fig. 1.**

**Fig. 2.**

**Fig. 3.**

**Fig. 4.**

**Fig. 5.**

**Fig. 6.**

| The numb. of generation | Time | The results of graphics in fig.1 | mean value | Dispersion |
|---|---|---|---|---|
| 1 | 0.118806 | 6 | 4.549 | 6.277 |
| 20 | 9.74541 | 36 | 55.726 | 66.295 |
| 40 | 22.0758 | 1 | 65.245 | 123.214 |
| 60 | 30.1153 | 22 | 46.415 | 71.488 |
| 80 | 39.7085 | 45 | 64.707 | 98.939 |
| 100 | 52.4288 | 4 | 100.861 | 134.168 |
| 120 | 62.9442 | 8 | 112.041 | 137.315 |
| 140 | 71.65 | 14 | 114.734 | 111.732 |
| 160 | 81.3515 | 20 | 102.106 | 97.489 |
| 180 | 90.9352 | 92 | 132.782 | 130.555 |
| 200 | 99.5314 | 213 | 12.203 | 122.492 |
| 220 | 109.667 | 22 | 73.505 | 79.951 |
| 240 | 118.557 | 11 | 43.961 | 45.932 |
| 260 | 128.221 | 1 | 43.387 | 56.352 |
| 280 | 133.987 | 22 | 58.961 | 58.282 |
| 300 | 145.564 | 4 | 66.400 | 79.9656 |
| 320 | 155.261 | 24 | 38.281 | 52.968 |
| 340 | 158.226 | 105 | 66.250 | 90.389 |
| 360 | 169.456 | 72 | 33.703 | 57.775 |
| 380 | 182.017 | 106 | 32.426 | 71.696 |

Tabl.1 (Non-normalized case)

| The numb. of generation | Time | The results of graphics in fig.4 | mean value | Dispersion |
|---|---|---|---|---|
| 1 | 0.118806 | 1 | 1.290 | 1.901 |
| 20 | 9.74541 | 0.05 | 0.560 | 0.747 |
| 40 | 22.0758 | 0.225 | 0.984 | 1.143 |
| 60 | 27.6953 | 0.0166667 | 0.31 | 0.299 |
| 80 | 39.7085 | 0.1 | 0.373 | 0.422 |
| 100 | 52.4288 | 0.12 | 0.431 | 0.607 |
| 120 | 62.9442 | 0.183333 | 0.352 | 0.537 |
| 140 | 71.65 | 0.378571 | 0.317 | 0.336 |
| 160 | 81.3515 | 0.01875 | 0.154 | 0.231 |
| 180 | 90.9352 | 0.0611111 | 0.260 | 0.371 |
| 200 | 99.5314 | 0.25 | 0.210 | 0.327 |
| 220 | 109.667 | 0.213636 | 0.160 | 0.381 |
| 240 | 118.557 | 0.0291667 | 0.404 | 0.807 |
| 260 | 128.221 | 0.0461538 | 0.421 | 0.707 |
| 280 | 133.987 | 0.0142857 | 0.472 | 0.798 |
| 300 | 145.564 | 0.0666667 | 0.717 | 1.023 |
| 320 | 155.261 | 0.11875 | 0.622 | 0.944 |
| 340 | 162.845 | 0.0117647 | 0.587 | 0.786 |
| 360 | 170.187 | 0.0583333 | 0.481 | 0.884 |
| 380 | 180.479 | 0.142105 | 0.167 | 0.612 |

Tabl.2 (Normalized case)

## References

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