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SUMMABILITY OF SERIES ON ROOT FUNCTIONS IN BOUNDARY VALUE PROBLEMS

Abstract

In the present paper we consider the homogeneous boundary problem for ordinary differential equations of the second order with variable coefficients when spectral parameter enter into the equation.

In many papers and monographs of many authors boundary value problems for ordinary differential equations are considered in case when principal parts of differential equations are with constant coefficients [1-7]. In particular in Danford N. and Shwartz I.T. [1] book and also in papers of Keselman G.M. [2] and Mikhaylov V.P. [3] basic properties of a system of root functions of regular boundary-value problems with constant coefficients of principal parts are studied on the sets of smooth functions in Khramov A.P. [4], Konstyuchenko A.G. and Radzievskiy T.V. [5], Kontyuchenko A.G. and Shkalikov A.A. [6]. In Rasulov M.L. [7] known book the expansion of smooth functions is studied in case when coefficients of principal part are complex-valued functions. But on roots of characteristic equation were imposed the strong restriction of the following form:

$$\arg \omega_j (x) = const \ \text{ and } \ \arg (\omega_j (x) - \omega_k (x)) = const$$

The generalization of theorem is considered in [8,9] for the system of the first order of ordinary differential equations.

In the present paper we consider the homogeneous boundary problem for ordinary differential equations of the second order with variable coefficients when spectral parameter linearly enter into the equation and boundary conditions:

$$L(\lambda) u = \lambda u(x) + e^{ic(x)} u''(x) + \frac{b(x)}{x^{\beta}} u(x) + u'(x_0) + \int_0^1 B(x,y) u''(y)\, dy = 0,$$

$$x \in [0,1], \quad x_0 \in (0,1) \tag{1}$$

$$L_1(\lambda) u = \lambda \left(u'(0) + u(1) + u'(x_1) + \int_0^1 T_1(x) u'(x)\, dx \right) + \int_0^1 T_2(x) u'(x)\, dx = 0,$$

$$L_2 u = u(1) + u(x_2) + \int_0^1 T_3(x) u(x)\, dx = 0. \tag{2}$$

Remember that non-trivial solution of the problem (1)-(2) from $W_2^2(0,1)$ is called eigenfunction of the problem (1)-(2). Let λ_0 be eigen number and $u_0(x)$ corresponding eigenfunction form $W_2^2(0,1)$. The problem (1)-(2) of the functions $u_k(x)$ are called associated functions of the rank k if they belong to $W_2^2(0,1)$ and they are the solutions of the following problem

$$L(\lambda_0)u_k + u_{k-1} = 0 \tag{3}$$

$$L_1(\lambda_0)u_k + C_1 u_{k-1} = 0,$$

$$L_2 u_k = 0,$$

where

$$C_1 u = u'(0) + u(1) + u'(x) + \int_0^1 T_1(x)\, u'(x)\, dx.$$

The eigen and associated functions are joined under the joint name called root functions.

Let $f \in H,\ H$ be a separable Hilbert space. Consider the series

$$\sum_s c_s(t)\, e_s.$$

Here $c_s(t)$ depends on t and it is determined by the following form. If e_s is eigenvector T, corresponding to the characteristic number λ_p has no associates then

$$c_s(t) = e^{-\lambda_p^\delta t} c_s$$

c_s is a coefficient of formal expansion of the vector f by the system $\{e_i\},\ \ i = 1, 2, ...$

If $e_s, ..., e_{s+j}$ form Jordan chain then $c_{s+i}(t)\ (i = 0, 1, ..., j)$ are expansion coefficients of residual of the subintegral function

$$\frac{1}{2\pi i}\int_{\gamma_p} e^{-\lambda^\delta t} T(I - \lambda T)^{-1} f d\lambda$$

by the vectors $e_s, ..., e_{s+j}$ where γ_p is closed contour containing only λ_p.

Definition. *We'll say that the series $\sum_s c_s(t)\, e_s$ corresponding to f is summable to k f by Abel method of the order δ if for this series there exists the subsequence of partial sums S_{N_v} converging in H for all $t > 0$ such that*

$$u(t) = \sum_{v=1}^{\infty}\left(\sum_{s=N_{v+1}}^{N_{v+1}} c_s(t)\, e_s\right),$$

$$\lim_{t\to 0} u(t) = f$$

Theorem. *Let the following conditions*

1. $c(x), b(x) \in C[0,1]$;

2. $B(x,y) \in C([0,1]\times[0,1])$;

3. $\beta < \frac{1}{2}, x_0 \in [0,1],\ x_1, x_2 \in (0,1)$;

4. at some $\theta \in \pi$, $\max |c(x)| < \theta$*;*

5. $T_1(x)$, $T_2(x)$, $T_3(x) \in L_2(0,1)$ *be fulfilled.*

Then spectrum of the problem (1)-(2) is discrete and system of vector-functions

$$\left\{ \left(u_k(x), u_k'(0) + u_k(1) + u_k'(x_1) + \int_0^1 T_1(x) u_k'(x) dx \right) \right\},$$

where $u_k(x)$ are root functions of the problem (1)-(2) form the Abel basis of the order $\alpha \in \left(\frac{1}{2}, \frac{\pi}{2\theta}\right)$ in the space $L_2(0,1) \oplus C$.

Proof. Consider the operator A determined by the equation

$$Au = e^{ic(x)} u''(x) + Bu(x), \tag{4}$$

where

$$Bu(x) = \frac{b(x)}{x^\beta} u(x) + u'(x) + \int_0^1 B(x,y) u''(y) dy.$$

In the Hilbert space $H = L_2(0,1) \oplus C$ we consider operators determined by the equations

$$D(oz) = \left\{ \vartheta / \vartheta = (u, C_1 u),\ u \in W_2^2(0,1),\ C_2 u = 0 \right\} \tag{5}$$

$$oz(u, C_1 u) = (-Au, Tu),$$

where

$$C_1 u = u'(0) + u(1) + u'(x_1) + \int_0^1 T_3(x) u'(x) dx,$$

$$C_2 u = u(1) + u(x_2) + \int_0^1 T_3(x) u(x) dx,$$

$$Tu = \int T_2(x) u'(x) dx.$$

If $u_0, u_1, ..., u_k$ are the root functions of the problem (1)-(2) i.e. the relation (3) is fulfilled then the vectors $\vartheta_1 = (u_1, C_1 u_1)$, $\vartheta_2 = (u_2, C_1 u_2), ..., \vartheta_k = (u_k, C_1 u_k)$ are root functions of the operator oz i.e. the relations

$$(\lambda I - oz) u_p + u_{p-1} = 0, \quad p = 0, 1, ..., k, \text{ where } \vartheta_{-1} = 0$$

are fulfilled. Thus if systems of root functions of the operator oz form Abel basis of the order a in the space $L_2(0,1) \oplus C$ then the system of the vector-functions $\{(u_k, C_1, u_k)\}$ where u_k are root functions of the problem (1)-(2) form Abel basis of the order a in the space $L_2(0,1) \oplus C$.

Evidently the linear manifold $\left\{ (u_1 \vartheta) \left| u \in W_2^2[0,1], C_2 u = 0,\ \vartheta = C_1 u \right. \right\}$ is dense in the Hilbert space $H = L_2(0,1) \oplus C$ i.e. the domain of determination of the operator oz is dense in $L_2(0,1) \oplus C$. By virtue of ([10], p.437) the compact embedding $W_2^2(0,1) \subset L_2(0,1)$ holds and relation of equivalence

$$S_j \left(J : W_2^2(0,1), L_2(0,1) \right) \sim J^{-2}, \quad j = 1, ..., \infty,$$

where J is an embedding operator from $W_2^2(0,1)$ in $L_1(0,1)$, S_j are $s-$numbers of embedding operator. For some $\lambda \in p(A)$ we have

$$S_j(J:H(oz),H) = S_j(J:H(\lambda_0 I - oz),H) = S_j(J:R(\lambda_0,oz):H,H) \leq$$

$$\leq \|R(\lambda_0,oz)\|_{B\left(L_2(0,1)\oplus C, W_2^2(0,1)\oplus C\right)} S_j\left(\tau : W_2^2(0,1)\oplus C, L_2(0,1)\oplus C\right) \leq$$

$$\leq c\left(S_j\left(J:W_2^2(0,1),L_2(0,1)\right) + S_j(J:c,c)\right) \leq cj^{-2}, \quad j=1,2,...,\infty,$$

i.e. the operator oz satisfies the condition: the embedding $H(oz) \subset H$ is compact and $S_j(J:H(A),H) \leq cj^{-2}$.

Instead of the equations

$$(\lambda I - oz)\vartheta = F, \;\; F(f,f_1) \in L_2(0,1) \oplus C \tag{6}$$

consider its equivalent system

$$L(\lambda)u = \lambda u + ozu = f, \tag{7}$$

$$L_1(\lambda)u = \lambda C_1 u + Tu = f_1$$

$$C_2 u = 0.$$

For all complex λ from the angle

$$\sup_{x\in[0,1]} c(x) + \varepsilon < \arg\lambda < 2\pi + \inf_{x\in[0,1]} c(x) - \varepsilon, \tag{8}$$

and sufficiently great by the module $|\lambda|$ the system (7) at $f \in L_2(0,1)$, $f_1 \in C$, has a unique solution $u \in W_2^2(0,1)$ and the estimation

$$\|u\|_{W_2^2(0,1)} + |\lambda|\left(\|u\|_{L_2(0,1)} + |C_1 u|\right) \leq C(\varepsilon)\left(\|f\|_{L_2(0,1)} + \|f_1\|\right). \tag{9}$$

is true.

Then equation (6) has solution of the form

$$\vartheta = (u, C_1 u).$$

Consequently by virtue of (9)

$$\|\vartheta\|_{L_2(0,1)\oplus C} \leq C\left(\|u\|_{L_2(0,1)} + |C_1 u|\right) = C(\varepsilon)|\lambda|^{-1}\|F\|_{L_2(0,1)\oplus C}.$$

Thus in the angle (8) we have

$$\|R(\lambda,oz)\| \leq C|\lambda|^{-1}, \;\; |\lambda| \to \infty.$$

We choose θ satisfying the inequality

$$\pi > \theta > \max\left\{-\min_{x\in[0,1]} c(x), \max_{x\in[0,1]} c(x)\right\}.$$

Then the angle

$$|\arg\lambda| \geq \theta, \tag{10}$$

is situated inside the angle (8). Thus all rays from the angle (10), i.e. $Z = re^{i\varphi}$, $\theta < \varphi < 2\pi - \theta$ are the rays of minimal increase of the resolvent $R(\lambda, oz)$.

Thus all the conditions of the theorem from ([11], p.345) are revised. According to this abstract theorem the root vectors of the operator oz are summed by Abel of the order $\alpha \in \left(\frac{1}{2}, \frac{\pi}{2\theta}\right)$ in the space $H = L_2(0,1) \oplus C$.

The theorem is proved.

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