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A MIXED PROBLEM FOR SOME CLASSES QUASILINEAR SOBOLEV TYPE EQUATION

Abstract

Existence, uniquenes and well-posendness for a some classes quasilinear Sobolev type equation are established.

Let Ω be a bonded domain of R^n with a sufficiently smooth boundary Γ. We consider the mixed problem for the higher order quasilinear hyperbolic equation in the cylinder $Q = (0,T) \times \Omega$.

$$u_{tt} + \alpha(-1)^k \Delta^k u_{tt} + a(t)(-1)^l \Delta^l u = f(t, x, \delta_{l-k} u), \tag{1}$$

with boundary conditions

$$\Delta^i u(t,x) = 0,\ (t,x) \in [0,T] \times \Gamma,\ i = 0, 1, ..., l-1, \tag{2}$$

and initial conditions

$$u(o,x) = u_0(x),\ u_t(o,x) = u_1(x),\ x \in \Omega, \tag{3}$$

where Δ-denotes the Laplacian;

$$u_{tt} = \frac{\partial^2 u}{\partial t^2},\ l \geq k,\ \delta_{l-k} u = \left(u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}, ..., \frac{\partial^{l-k} u}{\partial x_n^{l-k}} \right).$$

We will investigate the problem (1)-(3) at the following assumptions:

1^0. $\alpha > 0$;

2^0. Suppose $a(.) \in C^1[0,T]$ and $a(t) \geq a_0 > 0,\ t \in [0,T]$;

3^0. Suppose $F : [0,T] \times \overline{\Omega} \times R^{\varkappa} \to R$ is an element of the space $C^1([0,T] \times \overline{\Omega} \times R^{\varkappa})$, where $\varkappa = \dfrac{(n+l-k)!}{n!(l-k)!}$;

4^0. Suppose that for any $(t, x, \zeta) \in [0,T] \times \overline{\Omega} \times R^{\varkappa}$ the following estimations are satisfied [1]

$$|f(t,\varkappa,\zeta)| + |f_t(t,\varkappa,\zeta)| \leq c \left(\sum_{|\beta| < l-k-\frac{n}{2}} |\zeta_\beta| \right) \cdot \left(g(x) + \sum_{l-k-\frac{n}{2} \leq |\beta| \leq l-k} |\zeta_\beta|^{\frac{p(\beta)}{r}} \right), \tag{4}$$

$$\left| f_{\zeta_\gamma}(t,\varkappa,\zeta) \right| \leq c \left(\sum_{|\beta| < l-k-\frac{n}{2}} |\zeta_\beta| \right) \cdot \left(g_\gamma(x) + \sum_{l-k-\frac{n}{2} \leq |\beta| \leq l-k} |\zeta_\beta|^{\frac{q(\beta)}{r}} \right), \tag{5}$$

where $c(.) \in C(R_+; R_+)$ and

a) If $n > 4k$, then $r = \dfrac{2n}{n-4k}$, $g(.) \in L_{\frac{2n}{n-4k}}(\Omega)$, and

[1] If $l - k - \frac{n}{2} \leq 0$ then $c(\zeta) = c$ is constant

1. for $|\beta| = l-k-\frac{n}{2} : p(\beta) \in [1,\infty)$;

2. for $|\beta| > l-k-\frac{n}{2} : p(\beta) \le \frac{2n}{n-2(l-k-|\beta|)}$;

3. for $|\gamma| < l-k-\frac{n}{2} : g_\gamma(.) \in L_{\frac{2n}{n+4k}}(\Omega)$ and

in case $|\beta| = l-k-\frac{n}{2} : q(\beta) \in [1,\infty)$;

in case $|\beta| > l-k-\frac{n}{2} : q(\beta) \le \frac{2n}{n-2(l-k-|\beta|)}$;

4. for $|\gamma| = l-k-\frac{n}{2} : g_\gamma(.) \in L_p(\Omega),\ p \ge 1$ and

in case $|\beta| = l-k-\frac{n}{2} : q(\beta) \in [1,\infty)$,

in case $|\beta| > l-k-\frac{n}{2} : q(\beta) \le \frac{2n}{n-2(l-k-|\beta|)}$;

5. for $|\gamma| > l-k-\frac{n}{2} : g_\gamma(.) \in L_{\frac{n}{n+k-|\gamma|}}(\Omega)$, and

in case $|\beta| = l-k-\frac{n}{2} : q(\beta) \in [1,\infty)$;

in case $|\beta| > l-k-\frac{n}{2} : q(\beta) \le \frac{4n(k+l-|\gamma|)}{(n+4k)\cdot(n-2)(l-k-|\beta|)}$;

b) If $n \le 4$ then $r = 1,\ g(.) \in L_1(\Omega)$, and

1. for $|\beta| = l-k-\frac{n}{2} : p(\beta) \in [1,\infty)$;

2. for $|\beta| > l-k-\frac{n}{2} : p(\beta) \le \frac{2n}{n-2(l-k-|\beta|)}$;

3. for $|\gamma| < l-k-\frac{n}{2} : g_\gamma(.) \in L_1(\Omega)$ and

in case $|\beta| = l-k-\frac{n}{2} : q(\beta) \in [1,\infty)$,

in case $|\beta| > l-k-\frac{n}{2} : q(\beta) \le \frac{2n}{n-2(l-k-|\beta|)}$;

4. for $|\gamma| = l-k-\frac{n}{2} : g_\gamma(.) \in L_1(\Omega$, and

in case $|\beta| = l-k-\frac{n}{2} : q(\beta) \in [1,\infty)$,

in case $|\beta| > l-k-\frac{n}{2} : q(\beta) < \frac{n+4k}{n-(l-k-|\beta|)}$;

5. for $|\gamma| > l-k-\frac{n}{2} : g_\gamma(.) \in L_{\frac{n+2(l-k-|\gamma|)}{2n}}(\Omega)$, and

in case $|\beta| = l-k-\frac{n}{2} : q(\beta) \in [1,\infty)$,

in case $|\beta| > l-k-\frac{n}{2} : q(\beta) \le \frac{n+2(l-k-|\gamma|)}{n-2(l-k-|\beta|)}$.

We introduce the Hilbert space $\mathcal{H} = \hat{H}^{l-k} \times L_2(\Omega)$ with the following scalar products:

$$\left\langle w^1, w^2 \right\rangle = \int_\Omega \nabla^{l-k} u \cdot \nabla^{l-k} u_2 dx + \int_\Omega \upsilon_1 \cdot \upsilon_2 dx,$$

where $\nabla^s = \nabla^{\left[\frac{s}{2}\right]}$ if s -odd, $\nabla^s = \Delta^{\left[\frac{s}{2}\right]}$ if s- even,

$$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right), \hat{H}^s = \{u : u \in W_2^s(\Omega), \Delta^i u(x) = 0, x \in \Gamma,$$

$$s = 0, 1, ..., \left(\frac{r}{2}\right)\Big\},$$

$\left(\frac{r}{2}\right) = [\frac{r}{2}] - 1$, if r-odd and $\left(\frac{r}{2}\right) = [\frac{r}{2}]$ if r even.

We denote introduce also the Hilbert spaces $\mathcal{H}_0 = \hat{H}^{2(l-k)} \times \hat{H}^{l-k}$.

The main result of this paper is shown in the following theorem.

Theorem. *Let conditions $1^0 - 4^0$ be satisfied. Then for any $u_0 \in \hat{H}^{2(l-k)}$ and $u_1 \in \hat{H}^{(l-k)}$ the problem (1)-(3) has a unique solutions*

$$u \in C([0, T']; H^{2(l-k)}) \cap C^1([0, T^{'}]; \hat{H}^{l-k}) \cap C^2\left([0, T^{'}]; L_2(\Omega)\right)$$

where

$$T' \in (0, t_0),\ t_0 = \varphi(\|u_0\|_{\hat{H}^{l-k}} + \|u_1\|_{L^2(\Omega)}), \varphi(.) \in C(R_+; R_+).$$

If for the solution of the problem (1)-(3) the apriori estimation

$$E(t) = \|u_1(t)\|_{\hat{H}^{l-k}} + \left\|u^{'}(t)\right\|_{L^2(\Omega)} \le c(E(0)),\ c(.) \in C(R_+; R_+),$$

is satisfied, then $t_0 = T$, otherwise

$$\lim_{t \to t_0 - 0} E(t) = +\infty.$$

We determine the system of bilinear forms in space $\mathcal{H}$:

$$\left\langle w^1, w^2 \right\rangle_{\mathcal{H}_t} = \sum_{i=1}^{2s+1} \frac{(-1)^{i+1} a(t)}{\alpha^i} \int_\Omega \nabla^{l-ik} u_1 \cdot \nabla^{l-k} u_2 dx + \\ + \int_\Omega \upsilon_1 \cdot \upsilon_2 dx \quad , \tag{6}$$

where $s \ge \frac{l}{4k}$, $s \in N$.

Lemma 1. *For any $t \in [0, T]$ the bilinear form (6) is scalar product in $\mathcal{H}$ and they determine equivalent norms in $\mathcal{H}$.*

Proof. By determination

$$\|w\|_{\mathcal{H}_t}^2 = \sum_{r=0}^{s} \frac{a(t)}{\alpha^{2r+1}} \cdot \int_\Omega \left|\nabla^{l-(2r+1)k} u\right|^2 - \sum_{r=1}^{s} \frac{\alpha(t)}{\alpha^{2r}} \int_\Omega \left|\nabla^{l-2rk} u\right|^2 dx + \\ + \int_\Omega |\upsilon|^2 dx \quad , \tag{7}$$

where $w = \begin{pmatrix} u \\ \upsilon \end{pmatrix} \in \mathcal{H}$.

On the other hand

$$\int_\Omega \left|\nabla^{l-2rk}u\right|^2 dx = \int_\Omega \nabla^{l-(2r+1)k}u\nabla^{l-(2r-1)k}u\ dx \le$$

$$\le \frac{\alpha}{2}\int_\Omega \left|\nabla^{l-(2r+1)k}u\right|^2\ dx + \frac{1}{2\alpha}\int_\Omega \left|\nabla^{l-(2r-)k}u\right|^2\ dx.$$

Hence from above inequality we have that

$$-\sum_{r=1}^{s}\frac{a(t)}{\alpha^{2r}}\int_\Omega \left|\nabla^{l-2rk}u\right|^2 dx \ge -\sum_{r=1}^{s}\frac{a(t)}{2\cdot\alpha^{2r-1}}\int_\Omega \left|\nabla^{l-(2r-1)k}u\right|^2 dx-$$

$$-\sum_{r=1}^{s}\frac{a(t)}{2\alpha^{2r+1}}\int_\Omega \left|\nabla^{l-(2r+1)k}u\right|^2 dx =$$

$$= -\sum_{r=1}^{s-1}\frac{a(t)}{2\alpha^{2r+1}}\int_\Omega \left|\nabla^{l-(2r+1)k}u\right|^2 dx - \sum_{r=1}^{s}\frac{a(t)}{2\alpha^{2r+1}}\int_\Omega \left|\nabla^{l-(2r+1)k}u\right|^2 dx = \qquad (8)$$

$$= \frac{a(t)}{2\alpha}\int_\Omega \left|\nabla^{l-k}u\right|^2 dx - \sum_{r=1}^{s-1}\frac{a(t)}{\alpha^{2r+1}}\int_\Omega \left|\nabla^{l-(2r+1)k}u\right|^2 dx-$$

$$-\frac{a(t)}{2\cdot\alpha^{2s+1}}\int_\Omega \left|\nabla^{l-(2r+1)k}u\right|^2 dx.$$

From (7)-(8) follows that

$$\|w\|^2_{\mathcal{H}_t} \ge \frac{a(t)}{2\alpha}\int_\Omega \left|\nabla^{l-k}u\right|^2 dx + \frac{a(t)}{2\alpha^{2s+1}}\int_\Omega \left|\nabla^{l-(2s+1)k}u\right|^2 dx+$$

$$+\int_\Omega |\upsilon|^2 dx \ge \frac{a_0}{2\alpha}\int_\Omega^\Omega \left|\nabla^{l-k}u\right|^2 dx + \int_\Omega |\upsilon|^2 dx \ge \min\left(\frac{a_0}{2\alpha},1\right)\|w\|^2.$$

The estimate $\|w\|_{\mathcal{H}_t} \le \|w\|,\ c>0$ obtain is consequensely Holder's inequality. We are introducing the notation

$$w=\begin{pmatrix}\upsilon_1\\ \upsilon_2\end{pmatrix}, w_0=\begin{pmatrix}u_0(x)\\ u_1(x)\end{pmatrix}, M=\begin{pmatrix}1 & 0\\ 0 & 1+\alpha(-1)^k\Delta^k\end{pmatrix}, D(M)=\hat{H}^l\times\hat{H}^{2k},$$

$$D(N(t))=\begin{pmatrix}0 & 1\\ -a(t)(-1)^l\Delta^k & 0\end{pmatrix},\ D(N(t))=\hat{H}^{2l}\times\hat{H}^l,$$

$$\mathcal{F}(t,w)=\begin{pmatrix}0\\ f(t,x,\delta_{l-k}\upsilon_1)\end{pmatrix}.$$

By the substitution $\upsilon_1=u,\ \upsilon_2=u_t$ the problem (1)-(3) is reduced to the equivalent problem

$$Mw_t = N(t)w + \mathcal{F}(t,w)$$

$$w(0)=w_0.$$

We consider the boundary value problem.

$$u+\alpha(-1)^k\Delta^k u=h(x),\ x\in\Omega$$
$$\Delta^i u(x)=0,\ x\in\Gamma,\ i=0,1,...,k+r-1 \tag{9}$$

where $h\in\hat{H}^r$. It is known that the problem (9) has a unique solution $u\in\hat{H}^{r+2k}$ and the following coercive estimate holds

$$\|u\|_{\hat{H}^{r+2k}}\le c\,\|h\|_{\hat{H}^r,}\quad c>0.$$

Hence we can determine the following linear bounded operator $G_\alpha:\hat{H}^{r+2k}\to\hat{H}^r$, where $u=G_\alpha h$.

It is clear that for any $w\in D(L(t))$ the following inequality

$$\|L_0(t)w\|\le c\,\|w\|_{\mathcal{H}_0}$$

is satisfied, i.e. for every $t\in[0,T]$ the $L_0(t)$ is linear bounded operator from $\mathcal{H}_0$ to $\mathcal{H}$. Since $\overline{D(L_0(t))}\,|_{\mathcal{H}_0}$, then for every $t\in[0,T]$ we may extend the operator $L_0(t)$ to unbounded operator $L(t)$ with domain $\mathcal{H}_0$. Hence, the problem (1)-(3) have the form as Cauchy's problem in Holbert space $\mathcal{H}$:

$$w^{'}(t)=L(t)\,w(t)+F(t,w(t)), \tag{10}$$

$$w(0)=w_0. \tag{11}$$

Lemma 2. F*or any* $w\in\mathcal{H}$ *the function* $\varphi(t)=\|w\|^2_{\mathcal{H}_t}$ *is continuously differentiable and the following estimate is satisfied:*

$$\frac{d}{dt}\left(\|w\|^2_{\mathcal{H}_t}\right)\le c\,\|w\|^2. \tag{12}$$

Proof. From (5) it follows that for every $w=\begin{pmatrix}u\\v\end{pmatrix}\in\mathcal{H},\ \varphi(t)$ is differentiable and

$$\frac{d}{dt}\left(\|w\|^2_{\mathcal{H}_t}\right)=\sum_{i=1}^{25+1}\frac{a^{'}(t)}{\alpha^i}\int_\Omega\left|\nabla^{l-ik}u\right|^2dx\le$$
$$\le\max_{t\in[0,T]}\frac{\left|a^{'}(t)\right|}{\alpha^i}\sum_{i=1}^{2s+1}\int_\Omega\left|\nabla^{l-ik}u\right|^2dx. \tag{13}$$

On the other hand $\Delta^i u(x)=0,\ x\in\Gamma,\ i=0,1,...,\ l-k-1$ and therefore

$$\int_\Omega\left|\nabla^{l-ik}u\right|^2dx\le c_i\int_\Omega\left|\nabla^{l-ik}u\right|^2dx,\ c_i>0\ i=1,2,...,25+1. \tag{14}$$

Using (14) in (13) we obtain the following inequality

$$\frac{d}{dt}\left(\|w\|^2_{\mathcal{H}_t}\right)\le\ddot{c}\int_\Omega\left|\nabla^{l-k}u\right|^2dx\le\bar{c}\,\|w\|^2.$$

where $c = \max\limits_{\substack{t\in[0,T] \\ i=1,2,\dots,2s+1}} \dfrac{(2s+1)\cdot\left|a^{'}(t)\right| c}{\alpha^{i}}$, $\bar{c} = \max(\bar{c})$.

Lemma 3. *For every $t \in [0,T]$, the operator $L(t)$ generates strong continuous semigroup in the space $\mathcal{H}$.*

Proof. Let $w\begin{pmatrix} u \\ \upsilon \end{pmatrix} \in D(L) = \mathcal{H}_0$, then

$$\langle L(t)w, w\rangle_{\mathcal{H}_t} = \sum_{i=1}^{2s+1} \frac{(-1)^{i+1}a(t)}{\alpha^{'}} \int_\Omega \nabla^{l-ik}\upsilon\cdot\nabla^{l-ik} u\, dx - \tag{15}$$
$$-(-1)^l a(t)\int_\Omega G_\alpha \Delta^l u\cdot \upsilon dx.$$

Using resolvent equation, we transform the second term as follows:

$$J = (-1)^l a(t)\int_\Omega G_\alpha\Delta^l u\cdot\upsilon dx = \frac{(-1)^l a(t)}{\alpha}\int_\Omega (1-G_\alpha)\nabla^{l-k}u\cdot\upsilon\, dx =$$

$$= \frac{(-1)^l a(t)}{\alpha}\int_\Omega \nabla^{l-k}u\cdot\upsilon dx - \frac{(-1)^{l-2k}}{\alpha^2}\int_\Omega (1-G_\alpha)\Delta^{l-2k}u\cdot\upsilon\, dx =$$

$$= \dots = a(t)\sum_{i=1}^{2s+1}\frac{(-1)^{l-ik}}{\alpha^i}\cdot\int_\Omega \Delta^{l-ik}u\cdot\upsilon dx -$$

$$-\frac{(-1)^{l-(2s+1)}a(t)}{\alpha^{(2s+1)k}}\int_\Omega G_\alpha\Delta^{l-(2s+1)}u\cdot\upsilon dx.$$

Applying Greens formulas we obtain that

$$J = -a(t)\sum_{i=1}^{2s+1}\frac{(-1)^{l-ik}}{\alpha^i}\cdot\int_\Omega \nabla^{l-ik}u\cdot\nabla^{l-ik}\upsilon\, dx -$$
$$-\frac{a(t)\cdot(-1)^{l-(2s+1)k}}{\alpha^{2s+1}}\int_\Omega G_\alpha\Delta^{l-(2s+1)k}u\cdot\upsilon dx.$$

Taking it into account in (15) we have

$$\langle L(t)w, w\rangle_{\mathcal{H}_t} = \frac{(-1)^{l-(2s+1)}a(t)}{\alpha^{2s+1}}\cdot\int_\Omega G_\alpha\Delta^{l-(2s+1)k}u\cdot\upsilon dx.$$

Applying the Hōlder and Cauchy's inequality in above equality we get the following estimation:

$$\langle L(t)w, w\rangle_{\mathcal{H}_t} \le \sup_{0\le t\le T}\frac{|a(t)|}{\alpha^{2s+1}}\left\|G_\alpha\Delta^{l-(2w+1)k}u\right\|_{L_2(\Omega)}\cdot\|\upsilon\|_{L_2} \le$$

$$\le c_1\left(\left\|G_\alpha\Delta^{l-(2s+9)k}u\right\|^2_{L_2(\Omega)} + \|\upsilon\|^2_{L_2(\Omega)}\right) \le c_2\|w\|^6_{\mathcal{H}_t}.$$

Therefore, for any $t\in[0,U]$, $L(t) - c_7 I$, is a dissipative operator.

Now, we will prove that, for any $t \in [0,T]$, $L(t) - c_2 + I$ is invertible operator in $\mathcal{H}$, where I - is a unique operator.

It is eaey to see that

$$L(t) + I = \begin{pmatrix} 1 & 1 \\ (-1)^{l+1} a(t) G_\alpha \Delta^l & 1 \end{pmatrix}.$$

Let $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{H}$. We considgr the following equation

$$L(t)w + w = h, \tag{16}$$

where

$$w = \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} \in \mathcal{H}_0. \tag{17}$$

The problem (16)-(17) is equation to the following boundary value problem:

$$\upsilon_1 + \upsilon_2 = h_1, \tag{18}$$

$$(-1)^{l+1} a(t) G_\alpha \Delta^l \upsilon_1 + \upsilon_2 = h_2, \tag{19}$$

$$\Delta^i \upsilon_1(x) = 0,\ x \in \Gamma,\ i = 0, 1, ...,\ l - k - 1, \tag{20}$$

$$\Delta^j \upsilon_2(x) = 0,\ x \in \Gamma,\ j = 0, 1, ...,\ r, \tag{21}$$

where if $l-k$ is even $r = \left[\dfrac{l-k}{2}\right] - 1$, if $l-k$ is odd $r = \left[\dfrac{l-k}{2}\right]$.

Hence from (68)-(22) we have

$$\left.\begin{aligned} \upsilon_2 = h_1 - \upsilon_1, \\ \Delta^j \upsilon_2(x) = 0,\ x \in \Gamma,\ j = 0, 1, ...,\ r, \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} (-1)^{l+1} a(t) \Delta^l \upsilon_1 + (-1)^{k+1} \Delta^k \upsilon_1 - \upsilon_1 = g, \\ \Delta^i \upsilon_1(x) = 0,\ x \in \Gamma,\ i = 0, 1, ...,\ l - k - 1, \end{aligned}\right\} \tag{23}$$

where $g = h_2 + (-1)^k \Delta^k h_2 - h_1 - (-1)^k \Delta^k h_1$.

Since $h_1 \in L_2(\Omega)$, $h_2 \in \hat{H}^{l-k}$ therefore $g \in (\hat{H}^{2k})'$.

Using the theory of elliptic operators from (22) and (23) we obtain, that $\upsilon_1 \in \hat{H}^{2k(l-k)}$, $\upsilon_2 \in \hat{H}^{l-k}$. Therefore, for any $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{H}$, the problem (16)-(17) has a solution $w = \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} \in \mathcal{H}_6$.

Thus, $L(t) - c_5 I$ if maximal dissipative operator, that's why $L(t)$ generates strong continuous semigroup.

Using the expression for $L(t)$, we obtain the following results.

Lemma 4. *The operator function $L(t)$ is strong continuously differentiable.*

Lemma 5. *The mappbng $(t, w) \to F(t, w)$ is from $[0,T] \times \mathcal{H}$ in $\mathcal{H}$ and satisfies the local Lipschitz condition, i.e.*

$$\left\|F(t_1, w') - F(t_2, w^2)\right\| \le c(\left\|w^1\right\| + \left\|w^2\right\|) \cdot [|t_1 - t_2| + \left\|w^1 + w^2\right\|],$$

where $c(.) \in C(R_+, R_+)$.

Proof. Let $n > 4k$. Then $\hat{H}^{2k} \subset L_{r'} \subset L_2(\Omega)$, where $r' = \dfrac{2n}{n-4k}$. From here we have that

$$L_2(\Omega) \subset L_r(\Omega) \subset (\hat{H}^{2k})',$$

where $r' = \dfrac{2n}{n+4k}$. Since $G_\alpha$is a linear bounded operator from $L_2(\Omega)$ to $(\hat{H}^{2k})'$, therefore for any $t \in [0,T]$ and $w \in \mathcal{H}$, we have the following inequality:

$$\|F(t,w)\| = \|G_\alpha f(t,x,\delta_{l-k}\upsilon)\|_{L_2(\Omega)} \le c_1 \|f(t,x,\delta_{l-k}\upsilon_1)\|_{(\hat{H}^{2k})'} \le$$
$$\le c_2 \|f(t,x,\delta_{l-k}\upsilon_1)\|_{L_r(\Omega)}.$$

Later using (4) we have

$$\|F(t,w)\| \le c\left\{\int\limits_\Omega \left| c\left(\sum_{|\beta|<l-k-\frac{n}{2}} \left|D^\beta \upsilon_1\right|\right) \times \right.\right.$$

$$\times \left.\left.\left(g(x) + \sum_{l-k-|\beta|\le l-k} \left|D^\beta_{\upsilon_1}\right|^{\frac{p(\beta)}{r}}\right)\right|^r dx\right\}^{1/2} \le c_1 \sup_{x\in\bar{\Omega}} c\left(\sum_{|\beta|<l-k-\frac{n}{2}} \left|D^\beta_{\upsilon_1}\right|\right) \times \qquad (24)$$

$$\times \left[\|g\|_{L_r(\Omega)} + \sum_{l-k-\frac{n}{2}\le|\beta|\le l-k} \left(\int\limits_\Omega \left|D^\beta \upsilon_1\right|^{p(\beta)} dx\right)^{1/2}\right].$$

From imbedding theorem we obtain that at $|\beta| < l-k-\dfrac{n}{2}$: $D^\beta \upsilon_1 \subset C(\bar{\Omega})$; at $|\beta| = l-k-\dfrac{n}{2}$: $D^\beta \upsilon_1 \in L_p(\Omega)$, where $p \ge 1$ at $|\beta| > l-k-\dfrac{n}{2}$: $D^\beta \upsilon_1 \in L_{p(\beta)}(\Omega)$, where $p(\beta) \le \dfrac{2n}{n-2(l-k-|\beta|)}$.

Taking it into account (24) we get that

$$\|F(t,w)\| \le c(\|\upsilon_1\|_{\hat{H}^{1-k}}) \le c(\|w\|).$$

Let

$$t_1,\ t_2 \in [0,T],\ w^1 = \begin{pmatrix} u_1 \\ \upsilon_1 \end{pmatrix},\ w^2 = \begin{pmatrix} u_2 \\ \upsilon_2 \end{pmatrix} \in \mathcal{H},$$

then

$$\left\|F(t_1,w^1) - F(F(t_2,w^2)\right\| \le c\,\|f(t_1,x,\delta_{l-k}u_1) - f(t_2,x,\delta_{l-k}u_2)\|_{L_r(\Omega)} =$$

$$= c\left\{\int\limits_\Omega \left|\left(\int_0^1 f'_t(t_1+\tau\overline{\Delta}t_1, x, \delta_{l-k}(u_1+\tau\overline{\Delta}u_1))d\tau\right)\overline{\Delta}t_1 + \right.\right.$$

$$\left.\left. + \sum_{|\gamma|\le l-k}\left(\int_0^1 f'_{\zeta_\beta}(t_1+\tau\overline{\Delta}t_1, x, \delta_{\rho-k}(u_1+\tau\overline{\Delta}u_1))d\tau\right) D^\gamma(u_1-u_2)\right|^r dx\right\}^{1/2},$$

where $\overline{\Delta}t_1 = t_2 - t_1, \overline{\Delta}u_1 = u_2 - u_1$.

Thus, using 3^0 we have that

$$\left\|F(t_1,w^1)-F(t_2,w^2)\right\|\leq c\left\{c\left(\sum_{|\beta|<l-k-\frac{n}{2}}\left|D^\beta u_1+\tau D^\beta\overline{\Delta}u_1\right|\right)\times\right.$$
$$\times\left(g(x)+\sum_{l-k-\frac{n}{2}\leq|\beta|\leq l-k}\left|D^\beta u_1+\tau D^\beta\overline{\Delta}u_1\right|^{\frac{p(\beta)}{r}}\right)dx\times|t_2-t_1|+$$
$$+\sum_{|\gamma|\leq l-k}\int_\Omega c\left(\sum_{|\beta|\leq l-k-\frac{n}{2}}\left|D^\beta u_1+\tau D^\beta\overline{\Delta}u_1\right|\right)\times$$
$$\left.\times\left(g_\gamma(x)+\sum_{l-k-\frac{n}{2}\leq|\beta|\leq l-k}\left|D^\beta u_1+\tau D^\beta\overline{\Delta}u_1\right|^{\frac{q(\beta)}{r}}\right)D^\gamma\overline{\Delta}u_1\right|^2\Bigg\}^{1/r}\leq$$
$$\leq c_1\sup_{x\in\Omega}c\left(\sum_{|\beta|<l-k-\frac{n}{2}}\left(\left|D^\beta u_1\right|+\left|D^\beta u_2\right|\right)\right)\left\{\|g\|_{L_r(\Omega)}+\right.$$
$$+\sum_{l-k-\frac{n}{2}\leq|\beta|\leq l-k}\left[\int_\Omega\left|D^\beta u_1\right|^{p(\beta)}dx+\int_\Omega\left|D^\beta u_2\right|^{p(\beta)}dx\right]^{1/r}|t_2-t_2|+$$
$$+\sum_{|\gamma|<l-k-\frac{n}{2}}\sup_{x\in\overline{\Omega}}c\left(\sum_{|\beta|<l-k-\frac{n}{2}}\left(\left|D^\beta u_1\right|+\left|D^\beta u_2\right|\right)\right)\cdot\left[\|g_\gamma\|^r_{L_2(\Omega)}+\right.$$
$$\left.+\sum_{l-k-\frac{n}{2}\leq|\beta|\leq l-k}\left(\int_\Omega\left|D^\beta u_1\right|^{q(\beta)}dx+\int_\Omega\left|D^\beta u_2\right|^{q(\beta)}dx\right)\right]^{1/r}\sup_{x\in\overline{\Omega}}|D^\gamma(u_2-u_1)|+$$
$$+\sum_{|\gamma|\geq l-k-\frac{n}{2}}c\sup_{x\in\overline{\Omega}}c\left(\sum_{|\beta|<l-k-\frac{n}{2}}\left(\left|D^\beta u_1\right|+\left|D^\beta u_2\right|\right)\right)\left[\|g_\gamma\|^r_{L_2(\Omega)}+\right.$$
$$+\sum_{l-k-\frac{n}{2}\leq|\beta|\leq l-k}\left[\int_\Omega\left|D^\beta u_1\right|^{q(\beta)}|D^\gamma(u_2-u_1)|^r dx+\right.$$
$$\left.+\int_\Omega\left|D^\beta u_2\right|^{q(\beta)}|D^\gamma(u_2-u_1)|^r dx\right]\leq$$
$$\leq c\left(\|u_1\|_{\hat{H}^{l-k}}+\|u_2\|_{\hat{H}^{l-k}}\right)\left[|t_2-t_1|+\|u_2-u_1\|_{\hat{H}^{l-k}}\right]\leq$$
$$\leq c\left(\left\|w^1\right\|+\left\|w^2\right\|\right)\left[\left|t_2-t_1+\left\|w^1-w^2\right\|\right]\right..$$

Therefore, all conditions of solvability theorem ([1]-[3]) are satisfied for the problem (11)-(12). Using the result of [3] we obtain for any $w_0\in\mathcal{H}_0$ exists $t_0=\varphi(\|w_0\|)$ such that the problem (10)-(11) has a unique solution $w\in([0,T^{'}];\ \mathcal{H}_0)\cap C^1([0,T^{'}];\ \mathcal{H})$, where $T\in(0,t_0)$ and if for solution of (10)-(11) holds the apriory estimate

$$\|w(t)\|\leq c(\|w_0\|),\ c(.)\in C(R_+,R_+),$$

then $T'=t_0=T$, else

$$\lim_{t\to t_0-0}\|w(t)\|=+\infty.$$

Hence, we have corresponding result for the problem (1)-(3).

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