Nazila B. RASSOULOVA

ON ONE PROBLEM OF DYNAMICS OF RECTANGULAR PRISM

Abstract

In the paper the solution of one problem from dynamics of rectangular prism is given. Unlike the earlier considered problems in [1, 2] this problem is characterized by the existence of free lateral conditions, that visibly complicates the construction of solution. One original method for finding original by Laplace is suggested.

The present paper is a continuation of a series of papers devoted to the elastic rectangular beam dynamics [1, 2].

The exact analytical solution of dynamic problem of a rectangular beam is first obtained in [1]. Since the Lame system of equations is integrated difficultly involving free (single-type) boundaries, in the present paper [1] the lateral conditions are given in a mixed form.

In the following paper [2] the problem is a little complicated by the existence now of only two free lateral surfaces. In the same place the method which allows to get the solution of problems in any form of loading is suggested.

In the present paper unlike the previous ones [1, 2] it is assumed, that all lateral surfaces are free from efforts, but other conditions (front and initial) keep previous meaning. Thus the considered statement is formulated by the following initially-bounded problem of mathematical physics.

The Lame motion equation in the vector form:

$$\rho\frac{\partial^2\bar{U}}{\partial t^2} = (\lambda+\mu)\, grad\ div\ \bar{U} + \mu\Delta\bar{U} \qquad \bar{U} = \bar{U}\,(u, \upsilon, w) \tag{1}$$

are satisfied in the space occupied by the rectangular beam

$$-a \le x \le a \qquad -b \le y \le b; \quad z > 0 \quad for \quad t > 0\ .$$

For $t > 0$

$$\bar{U} = \dot{\bar{U}} = 0 \tag{2}$$

$$\left.\begin{array}{l} \sigma_z = \sigma_0\,(x, y)\, f\,(t) \\ u = 0 \\ \upsilon = 0 \end{array}\right\} \quad for \quad z = 0 \tag{3}$$

And finally:

$$\begin{array}{l} \sigma_{xx} = \sigma_{xy} = \sigma_{xz} = 0 \quad for \quad x = \pm a \\ \sigma_{yx} = \sigma_{yy} = \sigma_{yz} = 0 \quad for \quad y = \pm b \end{array} \tag{4}$$

where $\bar{U}$ is a displacement vector, $\{\sigma\}$ is stress tensor, λ, μ are the Lame coefficients, t is a time, ρ is a material density.

Following [1,2] the problem (1)-(4) can be reduced to the integrating of more simplified system:

$$\left.\begin{array}{l}(\lambda+2\mu)\,H_1\varphi=\mu q H_2\psi_2\\ H_2\psi_1=0\\ H_0H_2\psi_2=-\frac{\sigma_0(x,y)}{\mu}\bar{f}\,(p)\end{array}\right\}\qquad(1')$$

Here

$$H_i=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\left(q^2+\frac{p^2}{c_i^2}\right);\quad i=1,2$$
$$H_0=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-q^2;$$

are Helmholtz operators and the three functions φ,ψ_1,ψ_2 and are connected with double transformations (by the Laplace (p) and Fourier (q) operators) of the displacement function:

$$\begin{array}{c}\bar{u}_s=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi_1}{\partial y}-q\frac{\partial\psi}{\partial x};\\ \bar{\upsilon}_s=\frac{\partial\varphi}{\partial y}-\frac{\partial\psi_1}{\partial x}-q\frac{\partial\psi_2}{\partial y};\\ \bar{w}_c=q\varphi-\frac{\partial^2\psi_2}{\partial x^2}-\frac{\partial^2\psi_2}{\partial y^2}\ .\end{array}\qquad(5)$$

Here, on the left part the index S corresponds the sine transformation, and the index C – to cosine Fourier transformation.

For simplicity we accept, that $\sigma_0\,(x,y)=const=\sigma_0$ we'll write system (1) in the form

$$\left\{\begin{array}{l}H_0H_1\varphi=-\frac{\sigma_0 q}{\lambda+2\mu}\bar{f}\,(p)\\ H_0H_2\psi_2=-\frac{\sigma_0}{\mu}\bar{f}\,(p)\\ \psi_1\equiv 0\end{array}\right.\qquad(1^*)$$

The first equation of the system has the evident partial solution

$$\varphi=-\frac{\sigma_0}{\lambda+\mu}\ \frac{\bar{f}\,(p)}{q\left(\frac{p^2}{c_1^2}+q^2\right)}\qquad(6)$$

which originate the normal stress on the bounds $x=\pm a;\ y=\pm b$

$$\begin{array}{c}\sigma_{xx}^{\varphi}=\sigma_{yy}^{\varphi}=\frac{\lambda\sigma_0}{(\lambda+2\mu)\left(\frac{p^2}{c_1^2}+q^2\right)}q\bar{f}\,(p)\\ \sigma_{xy}^{\varphi}=\sigma_{xz}^{\varphi}=\sigma_{yz}^{\varphi}=0\ .\end{array}\qquad(7)$$

The general homogeneous solution of the second equation of system (1*) in the simplest form:

$$\psi_2^h=Ach\sqrt{\frac{p^2}{c_2^2}+q^2x}+Bch\sqrt{\frac{p^2}{c_2^2}+q^2y}\qquad(7^*)$$

can compensate stress (7) on the bounds $x=\pm a;\quad y=\pm b$ created by solution (6). For this we should define the constants A and B properly from the conditions

$$\begin{array}{ll}\sigma_{xx}^{\varphi}+\sigma_{xx}^{h}=0 & for\quad x=\pm a\\ \sigma_{yy}^{\varphi}+\sigma_{yy}^{h}=0 & for\quad y=\pm b\end{array}$$

we find $A = C(a)$, $B = C(b)$, where the function $C(l)$ is denoted as

$$C(l) = \frac{\lambda\sigma_0}{(\lambda+2\mu)} \frac{\bar{f}(p)}{\left(\frac{p^2}{c_1^2}+q^2\right) ch\sqrt{\frac{p^2}{c_2^2}+q^2}\cdot l} \tag{8}$$

However the solution ψ_2^h additionally creates the tangential stresses σ_{xz} and σ_{yz} for $x = \pm a$ and $y = \pm b$. As it is clarified only the solution of the form

$$\psi_2^* = \sum_{k=0}^{\infty} A_k \cos\left(\frac{1}{2}+k\right)\frac{\pi}{a}x + \sum_{k=0}^{\infty} B_k \cos\left(\frac{1}{2}+k\right)\frac{\pi}{b}y \tag{9}$$

can completely satisfy the boundary conditions (4) together with the collection of solutions (6) and (7*).

Therefore from all forms of partial solutions of the equation

$$H_0 H_2 \psi_2 = -\frac{\sigma_0}{\mu}\bar{f}(p)$$

we take that one which has the form (9).

The prime algebra satisfying the boundary conditions leads to the following expression for coefficients of the series (9):

$$\begin{aligned} A_k &= 2E_1 \frac{(-1)^k}{\pi\eta_k\left(\eta_{1k}^2+q^2+\frac{p^2}{c_2^2}\right)\left(\eta_{1k}^2+q^2\right)} \\ B_k &= 2E_2 \frac{(-1)^k}{\pi\eta_{2k}\left(\eta_{2k}^2+q^2+\frac{p^2}{c_2^2}\right)\left(\eta_{2k}^2+q^2\right)} \end{aligned} \tag{9*}$$

where:

$$\eta_{1k} = \left(\tfrac{1}{2}+k\right)\tfrac{\pi}{a};\quad \eta_{2k} = \left(\tfrac{1}{2}+k\right)\tfrac{\pi}{b};\quad E_1 = E_1(a);\quad E_2 = E_2(b).$$

$$E_i(l) = \frac{\frac{\pi}{2}\lambda\sigma_0}{(\lambda+2\mu)\mu}\frac{\bar{f}(p)}{\frac{p^2}{c_1^2}+q^2}\frac{p^2}{c_2^2}\frac{th\sqrt{\frac{p^2}{c_2^2}+q^2}\cdot l}{\sqrt{\frac{p^2}{c_2^2}+q^2}\sum\limits_{k=0}^{\infty}\frac{1}{\eta_{ik}^2+\left(\frac{p^2}{c_2^2}+q^2\right)}\frac{q^2-\eta_{ik}^2}{q^2+\eta_{ik}^2}},\quad i=1,\ 2 \tag{10}$$

Granting that the solution $\psi_2 = const$ doesn't generate any stresses in solid, the general solution of the posed initially-bounded problem in transformed surfaces is represented in the form:

$$\varphi = -\frac{\sigma_0}{\lambda+2\mu}\frac{\bar{f}(p)}{q\left(\frac{p^2}{c_1^2}+q^2\right)} \tag{11}$$

$$\psi_2 = Ach\sqrt{\frac{p^2}{c_2^2}+q^2}x + Bch\sqrt{\frac{p^2}{c_2^2}+q^2}y + \sum_{k=0}^{\infty}A_k\cos\eta_{1k}x + \sum_{k=0}^{\infty}B_k\cos\eta_{2k}y\ , \tag{12}$$

where A, B, A_k, B_k are the functions only of the transformed parameters p and q defined above by formulas (8) and (9*).

Now we'll find the inverse transformation of functions, constructing the general solutions (11)-(12). The longitudinal potential (11) in solid generates only axial movement of w^{φ}

$$w_c^{\varphi} = q\varphi = -\frac{\sigma_0}{\lambda+2\mu}\frac{\bar{f}(p)}{\frac{p^2}{c_1^2}+q^2}$$
$$w^{\varphi} = -\frac{\sigma_0}{\lambda+2\mu}\int_0^t f(t-\tau)\,H\left(\tau-\frac{z}{c_i}\right)d\tau \qquad (13)$$

(13) represents the plane longitudinal wave propagating from the end wall along the axis z with the velocity c_1.

The finding of inverse Laplace transformations of the solution of type (7*) is not difficult

$$\psi_2^h = Ach\sqrt{\frac{p^2}{c_1^2}+q^2x} + Bch\sqrt{\frac{p^2}{c_1^2}+q^2y}\ .$$

The expression of the form

$$\psi_2^h = \frac{\lambda\sigma_0 c_2^2}{(\lambda+2\mu)}\frac{\bar{f}(p)}{\frac{p^2}{c_1^2}+q^2}\frac{ch\frac{\nu(p)\,x}{c_2}}{\nu^2(p)\,ch\frac{\nu(p)\,a}{c_2}}, \qquad (14)$$

where $v(p) = \sqrt{p^2+c_2^2q^2}$ is transformed by means of Efgoss formula [3].

$$F_a[\nu(p)] = \frac{ch\frac{\nu(p)x}{c_2}}{\nu(p)ch\frac{\nu(p)a}{c_2}};\quad g(p) = \frac{1}{\nu(p)};$$
$$g(p)\,e^{-\tau\nu(p)} \underset{\circ}{=} {}^{\circ}J_0\left(c_2q\sqrt{t^2-\tau^2}\right)H(t-\tau) = g(t,\tau);$$
$$F_a[\nu_t(p)]\,g(p) \underset{\circ}{=} {}^{\circ}\int_0^t F_a(\tau)\,J_0\left(c_2q\sqrt{t^2-\tau^2}\right)d\tau;$$
$$F_a(t) = G_a\left(t-\frac{x}{a}\right)+G_a\left(t+\frac{x}{a}\right)\ ; \qquad (15)$$
$$G_a(t) = \frac{1}{2}H(t) + \sum_{k=0}^{\infty}\frac{(-1)^k\sin\left(k+\frac{1}{2}\right)\frac{\pi c_2 t}{a}}{\left(k+\frac{1}{2}\right)}\ .$$

If we take into account, that

$$\frac{\bar{f}(p)}{\frac{p^2}{c_1^2+q^2}} \underset{\circ}{=} {}^{\circ}c_1\int_0^t f(t-\tau)\,H\left(\tau-\frac{z}{a}\right)d\tau, \qquad (16)$$

the inverse transformation of the functions $\psi_2^{h_1}$ can be represented in the form of convolution of functions (15) and (16).

The question relatively to the second part of solution (7*) is solved analogously:

$$\psi_2^{h_2} = Bch\sqrt{\frac{p^2}{c_2^2}+q^2y}$$
$$F_b[\nu(p)]\,g(p) \underset{\circ}{=} {}^{\circ}\int_0^t\left[G_b\left(\tau-\frac{y}{a}\right)+G\left(\tau+\frac{y}{c_2}\right)\right]J_0\left(c_2q\sqrt{t^2-\tau^2}\right)d\tau\ . \qquad (17)$$

Now we consider the expressions of the type:

$$\sum_{k=0}^{\infty} A_k \cos \eta_{1k} = \tilde{E}_1 \sum_{k=0}^{\infty} \frac{(-1)^k \cos \eta_{ik}}{\left(\eta_{1k}^2 + q^2 + \frac{p^2}{c_2^2}\right)\left(\eta_{1k}^2 + q^2\right)}\ , \tag{18}$$

where

$$\tilde{E}_1 = \frac{\pi}{2} \frac{\lambda \sigma_0}{(\lambda + 2\mu)\,\mu} \frac{\bar{f}(p)}{\frac{p^2}{c_1^2} + q^2} \frac{p^2}{c_2^2} \frac{th\sqrt{\frac{p^2}{c_2^2} + q^2 a}}{\sqrt{\frac{p^2}{c_2^2} + q^2}\sum\limits_{m=0}^{\infty} \frac{1}{\eta_{1m}^2 + \left(\frac{p^2}{c_2^2} + q^2\right)}}, \tag{19}$$

which represent solutions (8) for the big values of p and q whose originals now will correspond the initial stage of loading.

Generally, allowing for the previous experience of finding inverse transformations of remainder part of solutions we can fix, that the only difficulties to achieve the aim is infinite sum in denominator of formula (19).

Really, the original from $\frac{1}{p} th \frac{pa}{c_2}$ is the function:

$$\frac{1}{p} th \frac{pa}{c_2} \ {}_\circ\!\!=^\circ H(t) + 2\sum_{n=1}^{\infty} (-1)^n H\left(t - \frac{2na}{c_2}\right)$$

changing the sign in each time interval

$$\frac{2(n-1)\,a}{c_2} < t < \frac{2na}{c_2}\ .$$

Let's we turn to the expression containing the infinite sum in a denominator: we substitute this sum for corresponding integral:

$$\frac{c_2^2}{c_2^2 \left(\frac{1}{2} + m\right)^2 \frac{\pi^2}{a^2} + \nu^2(p^2)} \approx \int_0^{\infty} \frac{c_2^2 dx}{c_2^2\left(\frac{1}{2} + x\right)\frac{\pi^2}{a^2} + \nu^2(p)} =$$

$$= \frac{c_2 a}{\nu(p)\,\pi}\left(\frac{\pi}{2} - \arctan \frac{c_2 \pi}{2\nu(p)\,a}\right) = \frac{c_2 a}{\nu(p)\,\pi}\left[\frac{\pi}{2} - \frac{i}{2}\ln \frac{1 - i\frac{c_2\pi}{2\nu(p)a}}{1 + i\frac{c_2\pi}{2\nu(p)a}}\right]$$

Now we consider the function:

$$R(p) = \tfrac{1}{2}\left[\pi - i\nu_1(p)\right]^{-1};$$
$$\nu_1(p) = \ln\left(1 - \tfrac{2ic_2\pi}{2pa + ic_4\pi}\right)\ .$$

The further process of finding the original $R(p)$ is given below, without comments:

$$e^{-\tau_1 \nu_1(p)} = \left[1 - \frac{2ic_i\pi}{2pa + ic_i\pi}\right]^{-\tau_1}; \quad -\frac{2ic_i\pi}{2pa + ic_i\pi} = \nu_2(p);$$

$$\nu_8(p)\,e^{-\tau_2\nu_2(p)} = -\frac{ic_2\pi}{a} J_0\left(i\sqrt{\frac{2ic_i\pi\tau_2 t}{a}}\right) e^{-\frac{i\pi c_i t}{2a}} = g_4(t, \tau_2)$$

$$F_2\left[\nu_2\left(p\right)\right]=\frac{\left[1+\nu_2\left(p\right)\right]^{-\tau_1}}{\nu_2\left(p\right)}$$

$$F_2\left(p\right)=\frac{\left(1+p\right)^{\tau_1}}{p}=\frac{1}{\Gamma\left(\tau_1\right)}\int\limits_0^t e^{-u}u^{\tau_1-1}du=\frac{1}{\Gamma\left(\tau_1\right)}\left[\Gamma\left(\tau_1\right)-\Gamma\left(\tau_1,t\right)\right]=F_2\left(t,\tau_1\right)$$

$$g_2\left(p\right)=\nu_2\left(p\right);\quad g_2\left(p\right)e^{-\tau_2\nu_2(p)}=g_2\left(t,\tau_2\right)$$

$$e^{-\tau_1\nu_1(p)}=F_2\left[\nu_2\left(p\right)\right]g_2\left(p\right)=\int\limits_0^\infty F_2\left(\tau_2\tau_1\right)g\left(t,\tau_2\right)d\left(\tau_2\right)=g_1\left(t,\tau_1\right)$$

$$F_1\left[\nu_1\left(p\right)\right]=\frac{1}{2}\left[\pi-i\nu_1\left(p\right)\right]^{-1};\quad F_1\left(p\right)=\frac{1}{\frac{\pi}{2}-\frac{i}{2}p}=2iH\left(t\right)e^{-\pi it};\quad g_i\left(p\right)=1;$$

$$g_2\left(p\right)e^{-\tau_1\nu_1(p)}=g_1\left(t,\tau_1\right);\ F_1\left(t\right)=2e^{-\pi it}H\left(t\right);$$

$$R\left(p\right)=F_1\left[\nu_2\left(p\right)\right]g_1\left(p\right)=2i\int\limits_0^\infty e^{-\pi i\tau_1}g_1\left(t,\tau_1\right)d\tau_1$$

$$J_0\left(\theta i\sqrt{i}\right)=ber_0\theta+ibei_0\theta;\quad \theta=\sqrt{\frac{2c_2\pi\tau_2 t}{a}};$$

$$R\left(p\right)\ {}_\circ\!\!=^\circ R\left(t\right)=\frac{c_2\pi}{a}\int\limits_0^\infty\int\limits_0^\infty e^{-i\pi\left(\frac{c_2t}{2a}-\tau_1\right)}\left(ber_0\theta+ibei_0\theta\right)F_2\left(\tau_2,\tau_1\right)d\tau_2d\tau_1\ .$$

Further, on the basis of tables given in [3] we can easily find the sine and cosine originals of Fourier transformations.

References

[1]. Rassoulova N.B. *The wave propagation in prismatic beam subjected to the axial forces.* Izv. ANRF, jur. "Mekh. of solid", pp.176-179. (Russian)

[2]. Rassoulova N.B. *On dynamics of bar of rectangular cross section.* Trans. of ASME, Journal of App. Mech., 2001, July, v.68, pp.662-666.

[3]. Ditkin V.A., Prudnikov A.P. *Integral transformations and operational calculus.* Moscow, 1961. (Russian)

Nazila B. Rassoulova
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 394 720(off.)

Received April 29, 2003; Revised November 24, 2003.
Translated by Javadova I.V.