**MECHANICS**

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# TO THE PROBLEM OF WAVE PROPOGATION IN ELASTIC MEDIUM AT MOVEMENT OF CYLINDER

**Abstract**

*We consider the problem on two-dimensional wave propagation in elastic medium under rigid cylinder at continuous movement. The solution of the problem is constructed by the Laplace-Karson transform method in the form of total elliptic integrals respondent the jump boundary condition with subsequent change of speed on boundary stipulated by obtained solution.*

We consider a two-dimensional problem on wave propagation in elastic medium. The solutions of some analytical problems by numerical methods are known. In this paper we construct an analytical solution on instant application of speed to the cylinder with its some subsequent change.

The displacements $u$ and $v$ in polar system of coordinates are represented in the following form [1]

$$u = \frac{\partial \varphi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$
$$v = \frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta} \tag{1}$$

where $r,\ \theta$ are polar coordinates; $\varphi$ and $\psi$ are functions satisfying the following wave equations

$$a^2 \Delta \varphi = \frac{\partial^2 \varphi}{\partial t^2}$$
$$a^2 \Delta \psi = \frac{\partial^2 \psi}{\partial t^2}$$

$a,\ b$ are "extension" and "distortion" wave velocities, $\Delta$ is a laplacian operator, $t$ is time. We can represent the solution of these wave equations in Laplace-Karson transform respondent to the movement of cylinder inclusion in the following form

$$\bar{\varphi}_1 = CK_1\left(\frac{pr}{a}\right); \quad \bar{\psi}_1 = DK_1\left(\frac{pr}{b}\right), \tag{2}$$

where $\bar{\varphi}_1 = \dfrac{\bar{\varphi}}{\cos\theta}, \bar{\psi}_1 = \dfrac{\bar{\psi}}{\cos\theta}$; $K_1$ is the first order Macdonald function; $p$ is a parameter of Laplace-Karson transform; $C$ and $D$ are quantities to be defined.

At movement of medium without breaking off the inclusion the boundary conditions give [2]

$$\frac{\partial\varphi_1}{\partial r}-\frac{\psi_1}{r}=-\frac{\partial\psi_1}{\partial r}+\frac{\varphi_1}{r}. \tag{3}$$

Putting solution (2) at $r=r_0$ in relation (3) we obtain

$$C=-f\left(p\right)\left[\frac{p}{b}K_0\left(\frac{pr_0}{b}\right)+\frac{2}{r_0}K_1\left(\frac{pr_0}{b}\right)\right]$$
$$C=f\left(p\right)\left[\frac{p}{b}K_0\left(\frac{pr_0}{a}\right)+\frac{2}{r_0}K_1\left(\frac{pr_0}{a}\right)\right] \tag{4}$$

where $f\left(p\right)$ is a function to be defined from the boundary condition.

Putting expression (4) to solution (2), then to relation (1) we have

$$\bar{u}_1=\frac{\bar{u}}{\cos\theta}=\ f\left(p\right)L, \tag{5}$$

where

$$L=\frac{p^2}{ab}K_0\left(\frac{pr}{b}\right)K_0\left(\frac{pr}{a}\right)+\frac{p}{br}K_1\left(\frac{pr_0}{b}\right)K_1\left(\frac{pr}{a}\right)+$$
$$+\frac{2}{a}\frac{p}{r_0}K_1\left(\frac{pr_0}{b}\right)K_0\left(\frac{pr}{a}\right)+\frac{2}{r_0r}K_1\left(\frac{pr_0}{b}\right)K_1\left(\frac{pr}{a}\right)-$$
$$-\frac{p}{ar}K_0\left(\frac{pr_0}{a}\right)K_1\left(\frac{pr}{b}\right)-\frac{2}{r_0r}K_1\left(\frac{pr_0}{a}\right)K_1\left(\frac{pr}{b}\right).$$

On the boundary $r=r_0$ from relation (5) it follows

$$\bar{u}_1=pf\left(p\right)\left[\frac{p}{ab}K_0\left(\frac{pr_0}{a}\right)K_0\left(\frac{pr_0}{b}\right)+\frac{1}{br_0}K_0\left(\frac{pr_0}{b}\right)K_1\left(\frac{pr_0}{a}\right)+\right.$$
$$\left.+\frac{1}{ar_0}K_0\left(\frac{pr_0}{a}\right)K_1\left(\frac{pr_0}{b}\right)\right]. \tag{6}$$

Resorting to asymototic approximations of Macdonalds functions by defining the function $f$ from boundary condition, which leads to approximated fulfilment of boundary condition, we have exact solutions of wave equation in the domain of wave propagation respondent continuous movement of medium without breaking off the inclusion. We give the boundary condition

$$\left.u_{1t}\right|_{r=r_0}=H\left(t\right)V_0, \tag{7}$$

where $V_0$ is constant speed of inclusion; $H\left(t\right)$ is a unit Heaviside function.

Taking in solution (6)

$$p\to\infty;\quad K_0\left(z\right)\approx\sqrt{\frac{\pi}{2z}}e^{-z};\quad K_1\left(z\right)\approx\sqrt{\frac{\pi}{2z}}e^{-z}$$

and putting in (7), we have

$$V_0 = \frac{p^2}{ab} f(p) \sqrt{\frac{\pi b}{2pr_0}} \sqrt{\frac{\pi a}{2pr_0}} \left( p + \frac{a+b}{r_0} \right) e^{-\frac{pr_0}{a}} e^{-\frac{pr_0}{b}}.$$

Whence we define

$$f = 2r_0 \sqrt{ab} e^{\frac{pr}{a}} e^{\frac{pr}{b}}.$$

Substituting the obtained expression for $f$ in relation (5) and allowing for $\bar{u}_{1t} = p\bar{u}_1$ we obtain

$$\bar{u}_{1t} = \frac{2r_0 \sqrt{ab}}{\pi \left( p + \frac{a+b}{r_0} \right)} V_0 e^{\frac{pr_0}{a}} e^{\frac{pr_0}{b}} L. \tag{8}$$

On the boundary $r = r_0$

$$\bar{u}_{1t} = \frac{2r_0 \sqrt{ab}}{\pi \left( p + \frac{a+b}{r_0} \right)} V_0 e^{\frac{pr_0}{a}} e^{\frac{pr_0}{b}} \left[ \frac{p}{ab} K_0 \left( \frac{pr_0}{a} \right) K_0 \left( \frac{pr_0}{b} \right) + \right.$$
$$\left. + \frac{1}{br_0} K_0 \left( \frac{pr_0}{b} \right) K_1 \left( \frac{pr_0}{a} \right) + \frac{1}{ar_0} K_0 \left( \frac{pr_0}{a} \right) K_1 \left( \frac{pr_0}{b} \right) \right]. \tag{9}$$

Further, the originals of six members in expression (8) is to be defined.

Allowing for originals

$$pK_0 \left( \frac{pr}{c} \right) \to \frac{H\left(t - \frac{r}{c}\right)}{\sqrt{t^2 - \left( \frac{r}{c} \right)^2}}; \quad K_1 \left( \frac{pr}{c} \right) \to \frac{c}{r} \sqrt{t^2 - \left( \frac{r}{c} \right)^2};$$

$$K_1 \left( \frac{pr}{c} \right) \to \frac{c}{r} \frac{t}{\sqrt{t^2 - \left( \frac{r}{c} \right)^2}}$$

we can determinate [3]

$$pK_0 \left( \frac{pr_0}{b} \right) K_0 \left( \frac{pr_0}{a} \right) e^{\frac{pr_0}{a}} e^{\frac{pr_0}{b}} \to A_1(a,b) =$$

$$= \int_{\frac{r-r_0}{a}}^{t} \frac{d\tau}{\sqrt{\left( \left( t - \tau + \frac{r_0}{b} \right)^2 - \left( \frac{r_0}{b} \right)^2 \right) \left( \left( \tau + \frac{r_0}{a} \right)^2 - \left( \frac{r_0}{a} \right)^2 \right)}} = \tag{10}$$

$$= \frac{2H\left( t - \dfrac{r - r_0}{a} \right) F(K(a,b))}{m(a,b)},$$

where $F$ is the first order total elliptic integral

$$K(a,b)=\sqrt{\frac{\left(t-\dfrac{r-r_0}{a}\right)\left(t+2\dfrac{r_0}{b}+\dfrac{r+r_0}{a}\right)}{\left(t+2\dfrac{r_0}{b}-\dfrac{r-r_0}{a}\right)\left(t+\dfrac{r+r_0}{a}\right)}}$$

$$m(a,b)=\sqrt{\left(t+2\frac{r_0}{b}-\frac{r-r_0}{a}\right)\left(t+\frac{r+r_0}{a}\right)}$$

$$K_0\left(\frac{pr_0}{b}\right)K_1\left(\frac{pr_0}{a}\right)e^{\frac{pr_0}{b}}e^{\frac{pr_0}{a}}\to A_2(a,b)=$$

$$=\frac{a}{r}\int\limits_{\frac{r-r_0}{a}}^{t}\frac{\left(\tau+\dfrac{r_0}{a}\right)d\tau}{\sqrt{\left(\left(t-\tau+\frac{r_0}{b}\right)^2-\left(\frac{r_0}{b}\right)^2\right)\left(\left(\tau+\frac{r_0}{a}\right)^2-\left(\frac{r_0}{a}\right)^2\right)}}=$$

$$=\frac{r}{r_0}\int\limits_{\frac{r-r_0}{a}}^{t}\frac{d\tau}{\sqrt{\left(\left(t-\tau+\frac{r_0}{b}\right)^2-\left(\frac{r_0}{b}\right)^2\right)\left(\left(\tau+\frac{r_0}{a}\right)^2-\left(\frac{r_0}{a}\right)^2\right)}}+$$

$$+\frac{a}{r}\int\limits_{\frac{r-r_0}{a}}^{t}\frac{\tau d\tau}{\sqrt{\left(\left(t-\tau+\frac{r_0}{b}\right)^2-\left(\frac{r_0}{b}\right)^2\right)\left(\left(\tau+\frac{r_0}{a}\right)^2-\left(\frac{r_0}{a}\right)^2\right)}}=$$

$$=\frac{r_0}{r}\frac{2HF(k(a,b))}{m(a,b)}+\frac{2Ha}{rm(a,b)}\left(\frac{2r}{a}\Pi(n(a,b),k(a,b))-\frac{r+r_0}{a}F(k(a,b))\right)=$$

$$=\frac{2H\left(t-\dfrac{r-r_0}{a}\right)\left(2\Pi(n(a,b),k(a,b))-F(k(a,b))\right)}{m(a,b)}, \qquad (11)$$

where $\Pi$ is the third order total elliptic integral,

$$n(a,b)=\frac{t-\dfrac{r-r_0}{a}}{t+\dfrac{r+r_0}{a}},$$

$$pK_1\left(\frac{pr_0}{b}\right)K_0\left(\frac{pr_0}{a}\right)e^{\frac{pr_0}{b}}e^{\frac{pr_0}{a}}\to A_3(a,b)=$$

$$=\frac{b}{r_0}\int\limits_{\frac{r-r_0}{a}}^{t}\frac{\left(t-\tau+\frac{r_0}{b}\right)d\tau}{\sqrt{\left(\left(t-\tau+\frac{r_0}{b}\right)^2-\left(\frac{r_0}{b}\right)^2\right)\left(\left(\tau+\frac{r_0}{a}\right)^2-\left(\frac{r_0}{a}\right)^2\right)}}=$$

$$=\left(\frac{bt}{r_0}+1\right)\int\limits_{\frac{r-r_0}{a}}^{t}\frac{d\tau}{\sqrt{\left(\left(t-\tau+\frac{r_0}{b}\right)^2-\left(\frac{r_0}{b}\right)^2\right)\left(\left(\tau+\frac{r_0}{a}\right)^2-\left(\frac{r_0}{a}\right)^2\right)}}-$$

$$-\frac{b}{r_0}\int\limits_{\frac{r-r_0}{a}}^{t}\frac{\tau d\tau}{\sqrt{\left(\left(t-\tau+\frac{r_0}{b}\right)^2-\left(\frac{r_0}{b}\right)^2\right)\left(\left(\tau+\frac{r_0}{a}\right)^2-\left(\frac{r_0}{a}\right)^2\right)}}=$$

$$=\left(\frac{bt}{r_0}+1\right)\frac{2HF(k(a,b))}{m(a,b)}-\frac{2bH}{r_0m(a,b)}\times$$
$$\times\left(\frac{2r}{a}\Pi(n(a,b),k(a,b))-\frac{r+r_0}{a}F(k(a,b))\right)= \tag{12}$$

$$=\frac{2H\left(t-\frac{r-r_0}{a}\right)\left(\frac{bt}{r_0}+\frac{rb}{r_0a}+\frac{b}{a}+1\right)F(k(a,b))-2\frac{rb}{r_0a}\Pi(n(a,b),k(a,b))}{m(a,b)},$$

$$pK_1\left(\frac{pr_0}{a}\right)K_0\left(\frac{pr_0}{b}\right)e^{\frac{pr_0}{a}}e^{\frac{pr_0}{b}}\to A_4(a,b)=$$
$$=\frac{ab}{r_0r}\int\limits_{\frac{r-r_0}{a}}^{t}\sqrt{\left(\left(t-\tau+\frac{r_0}{a}\right)^2-\left(\frac{r_0}{a}\right)^2\right)\left(\left(\tau+\frac{r_0}{b}\right)^2-\left(\frac{r_0}{b}\right)^2\right)}d\tau. \tag{13}$$

Allowing for

$$\frac{p}{p+\frac{a+b}{r_0}}\to H(t)e^{-\frac{a+b}{r_0}t};\quad \frac{p}{p+\frac{a+b}{r_0}}\bar{A}\to\int\limits_0^t e^{-\frac{a+b}{r_0}(t-\tau)}A(\tau)d\tau,$$
$$\frac{p^2}{p+\frac{a+b}{r_0}}\bar{A}\to A-\frac{a+b}{r_0}\int\limits_0^t e^{-\frac{a+b}{r_0}(t-\tau)}A(\tau)d\tau, \tag{14}$$

denoting $e^{\frac{a+b}{r_0}t}$, we finally obtain the speed expression

$$u_t = \frac{2r_0 V_0 \sqrt{ab}}{\pi}\left[\frac{1}{ab}\left(A_1(a,b) - \frac{a+b}{r_0\mu}\int_{\frac{r-r_0}{a}}^{t} A_1(a,b)\,\mu d\tau\right) + \right.$$

$$+\frac{1}{br\mu}\int_{\frac{r-r_0}{a}}^{t} A_2(a,b)\,\mu d\tau + \frac{2}{ar_0\mu}\int_{\frac{r-r_0}{a}}^{t} A_3(a,b)\,\mu d\tau +$$

$$+\frac{2}{r_0 r}\left(A_4(a,b) - \frac{a+b}{r_0\mu}\int_{\frac{r-r_0}{a}}^{t} A_4(a,b)\,\mu d\tau\right) - \frac{1}{ar\mu}\int_{\frac{r-r_0}{b}}^{t} A_2(b,a)\,\mu d\tau -$$

$$\left. -\frac{2}{r_0 r}\left(A_4(b,a) - \frac{a+b}{r_0\mu}\int_{\frac{r-r_0}{b}}^{t} A_4(b,a)\,\mu d\tau\right)\right]. \tag{15}$$

At $r = r_0$ the total elliptic integrals $F$ and $\Pi$ contained in the expressions of the quantities $A_1$ and $A_2$ are connected with the relations

$$F(a,b) = F(b,a)$$

$$A_2^0(b,a) = \frac{2}{m}\left[2\Pi(n(b,a),k) - F(k)\right] = A_3^0(a,b) = \tag{16}$$

$$= \frac{2}{m}\left[\left(\frac{bt}{r_0} + 2\frac{b}{a} + 1\right)F(k) - 2\frac{b}{a}\Pi(a,b)\right].$$

Subject to (16), solution (15) gets the form

$$u_t = \frac{2r_0 V_0 \sqrt{ab}}{\pi}\left[\frac{1}{ab}\left(A_1^0 - \frac{a+b}{r_0\mu}\int_0^t A_1 \mu d\tau\right) + \right.$$
$$+\frac{1}{br_0\mu}\int_0^t A_2^0(a,b)\,\mu d\tau + \frac{1}{ar_0\mu}\int_0^t A_2^0(b,a)\,\mu d\tau, \tag{17}$$

where

$$A_1^0 = \frac{2F(k)}{m}$$

$$k = k(a,b) = k(b,a) = \sqrt{\frac{t\left(t + 2\frac{r_0}{b} + 2\frac{r_0}{a}\right)}{\left(t + 2\frac{r_0}{b}\right)\left(t + 2\frac{r_0}{a}\right)}}$$

$$m = m(a,b) = m(b,a) = \sqrt{\left(t + 2\frac{r_0}{b}\right)\left(t + 2\frac{r_0}{a}\right)}$$

$$n(a,b) = \frac{t}{t + 2\frac{r_0}{a}}, \quad n(a,b) = \frac{t}{t + 2\frac{r_0}{b}}$$

At calculations the quantities $k$ and $n$ may be close to unit which leads to great values of elliptic operators and here the table can't be used. Therefore, it is necessary to use the asymptotic formulae. We derive the asymptotic formula for $n \sim -1$

$$\Pi\left(\frac{\pi}{2}, n, k\right) \approx \Pi\left(\frac{89}{180}\pi, n, k\right) + \Phi(n,k) \tag{18}$$

at $n \to \infty$ or $k \to 1$, where

$$\Phi(n,k) =$$

$$= \frac{1}{\sqrt{1-n}\sqrt{k^2-n}}\left[\ln\frac{\frac{\pi}{180}\sqrt{\frac{k^2-n}{nk^2}} + \sqrt{\frac{1-n}{n}\left(\frac{\pi}{180}\right)^2 + \frac{1-k^2}{k^2}}}{\sqrt{\left(\frac{\pi}{180}\right)^2 + \frac{1-n}{n}}} - \ln\sqrt{\frac{1-k^2}{k^2}}\right].$$

Thus, the integrals containing singularities in integrand functions were reduced to elliptic integrals.

For determination of circumferential speed vector component $v_{1t}$ in the last solution it is necessary to permute the quantities $a$ and $b$.

The construction of solution of the given problem responds to jump boundary condition with subsequent change of speed at time. The change of spped at time on the boundary $a = 2000$ $m/\sec$ at $b = 1400$ $m/\sec$ is shown on Fig.1. Analogously to the Duhamel principle from the obtained solutions we can formulate the solution respondent to boundary conditions close to real conditions.

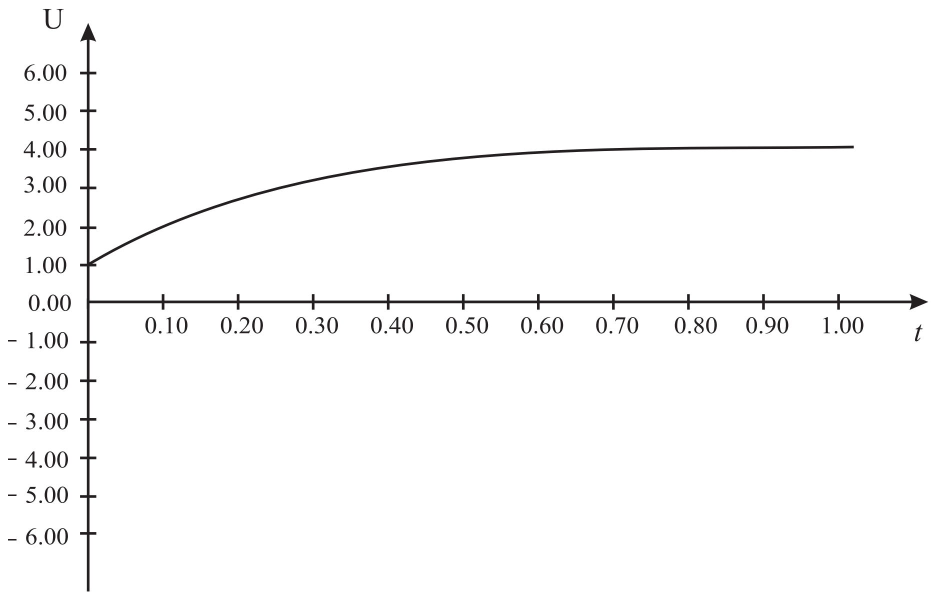


**Fig. 1.**

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