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ON CORRECT SOLVABILITY OF A LINEAR NONLOCAL PROBLEM OF GENERAL FORM FOR A CLASS OF THIRD ORDER HYPERBOLIC EQUATIONS

Abstract

In this paper we consider a nonlocal problem for a class of third order linear hyperbolic equations with measurable, in general, nonsmooth coefficients satisfying some p - integrability and boundedness conditions. The sufficient conditions for everywhere correct solvability of the considered problem are found.

Let the following equation be given

$$(V_0u)(t,x) \equiv D_tD_x^2u(t,x) + \sum_{\substack{i+j<3\\ i=0,1;\ j=0,1,2}} \left(D_t^iD_x^j(t,x)\right)A_{i,j}(t,x) = z_{1,2}(t,x), \tag{1}$$

$$(t,x) \in G = (t_0,t_1)\times(x_0,x_1)$$

under the initial condition

$$(l_0u)(x) \equiv u(t_0,x) = z_0(x), \quad x \in (x_0,x) \tag{2}$$

and boundary conditions

$$\begin{gathered}(l_1u)(t) \equiv u_t(t,x_0)\alpha_{1,1} + u_{tx}(t,x_0)\alpha_{1,2}+\\ +u_t(t,x_1)\beta_{1,1} + u_{tx}(t,x_1)\beta_{1,2} = z_1(t),\\ (l_2u)(t) \equiv u_t(t,x_0)\alpha_{2,1} + u_{tx}(t,x_0)\alpha_{2,2}+\\ +u_t(t,x_1)\beta_{2,1} + u_{tx}(t,x_1)\beta_{2,2} = z_2(t), \quad t\in(t_0,t_1).\end{gathered} \tag{3}$$

Here $u(t,x) = (u_1(t,x),...,u_n(t,x))$ is n - dimensional unknown vector-function; α_{ij} and β_{ij} are the given $n\times n$ dimensional constant matrices, $A_{i,j}(t,x)$ are $n\times n$ order matrix functions measurable on G, satisfying the conditions $A_{0,j}(t,x) \in L_p(G)$, $j=0,1$, and there exist the functions $A^0_{1,j}(t,x) \in L_p(x_0,x_1)$, and $A^0_{0,2}(x) \in L_p(t_0,t_1)$ such that the conditions

$$\|A_{1,j}(t,x)\|_{n\times n} \le A^0_{1,j}(x), \quad j=0,1$$

and

$$\|A_{0,2}(t,x)\|_{n\times n} \le A^0_{0,2}(x)$$

are satisfied almost everywhere on G, where $\|\cdot\|_{n\times n}$ is Euclidean norm of the corresponding matrix.

With respect to the right hand side of the equation and initial-boundary conditions we assume that $z_0(x) \in W^{(2)}_{p,n}(x_0,x_1)$ and $z_i(t) \in L_{p,n}(t_0,t_1)$, where $L_{p,n}(t_0,t_1)$

and $L_{p,n\times n}(t_0,t_1)$ are spaces of n dimensional row vectors and space of $n\times n$ matrices with elements from $L_p(t_0,t_1)$, respectively, $W_{p,n}^{(2)}(x_0,x_1)$ is a space of all vector functions with elements of $W_p^{(2)}(x_0,x_1)$.

Under these conditions we seek the solution of problem (1)-(3) in the S.L.Sobolev space

$$W_{p,n}\equiv W_{p,n}^{(1,2)}(G)=\left\{u\in L_{p,n}(G)\left|D_t^iD_x^ju\in L_{p,n}(G)\right.,\ i=0,1,\ j=0,1,2\right\}$$

with the dominating derivative $D_tD_x^2$.

The sets of values of the operators l_0,l_1 and l_2 defined on $W_{p,n}$ are disconnected sets. It means that when equation (1) is considered at initial boundary conditions of form (2) and (3) it is not necessary to impose accordance type additional conditions on the right hand side of these conditions. By this conditions (2) and (3) are more natural than classic type initial-boundary condition [1,2,3].

Representation of the functions $u\in W_{p,n}$ by means of defining operators.

Consider (1), (2), (3). We write this problem in the operator form

$$Vu=z, \tag{4}$$

where

$$V=(V_0,l_0,l_1,l_2),$$

$$z=(z_{1,2}(t,x),\ z_0(x),\ z_1(t),\ z_2(t))\in E_{p,n},$$

$$E_{p,n}=L_{p,n}(G)\times W_{p,n}^{(2)}(x_0,x_1)\times L_{p,n}(t_0,t_1)\times L_{p,n}(t_0,t_1).$$

We assume the norm in the space $E_{p,n}$ be defined in the natural form with the help of the equality

$$\|z\|_{E_{p,n}}=\|z_{1,2}\|_{L_{p,n}(G)}+\|z_0\|_{W_{p,n}^{(2)}(x_0,x_1)}+\|z_1\|_{L_{p,n}(t_0,t_1)}+\|z_2\|_{L_{p,n}(t_0,t_1)}.$$

It is easily proved that the operator $V:W_{p,n}\to E_{p,n}$ is linear and bounded.

If for any $z\in E_{p,n}$ equation (4) (problem (1), (2), (3)) has a unique solution $u\in W_{p,n}$ and

$$\|u\|_{W_{p,n}}\le M\|z\|_{E_{p,n}} \tag{5}$$

then we shall say that the operator V of equation (4) realizes homomorphism between $W_{p,n}$ and $E_{p,n}$ or we shall say that problem (1), (2), (3) is everywhere correct solvable, where $M>0$ is a constant (depending on coefficients of boundary value problem (1), (2), (3)) independent of $z\in E_{p,n}$.

We investigate problem (1), (2), (3) with help of integral representations of special form for the functions $u\in W_{p,n}$. For the function $u\in W_{p,n}$ we can find different integral representations. We can represent the functions $u\in W_{p,n}$, for example, in the form

$$u(t,x)=u(t_0,x)+\int_{t_0}^{t}\left[u_t(\tau,x_0)-(x-x_0)u_{tx}(\tau,x_0)\right]d\tau+$$

$$+\int_{t_0}^{t}\int_{x_0}^{x}(x-\xi)\,u_{txx}(\tau,\xi)\,d\tau d\xi \tag{6}$$

by the traces $u(t_0,x)$, $u_t(t,x_0)$, $u_{tx}(t,x_0)$ and dominating derivative $u_{txx}(t,x)$ ([4]).

Now we put such a question. Can we find a representation that would admit to determine the function $u(t,x)$ uniquely with respect to values $(l_0u)(x)$, $(l_1u)(t)$, $(l_2u)(t)$ and $D_tD_x^2u(t,x)$ for the function $u\in W_{p,n}$? For studying this question we use formula (6). At first we write (6) in the following form

$$u(t,x)=(Qb)(t,x)\equiv\varphi(x)+\int_{t_0}^{t_1}\left[b_{1,0}(\tau)+(x-x_0)\,b_{1,1}(\tau)\right]d\tau+$$
$$+\int_{t_0}^{t}\int_{x_0}^{x}(x-\xi)\,b_{1,2}(\tau,\xi)\,d\tau d\xi,\quad (t,x)\in G, \tag{7}$$

where

$$b=\left(b_{1,2}(t,x),\varphi(x),b_{1,0}(t),b_{1,1}(t)\right)$$

and

$$\varphi(x)=u(t_0,x),\ b_{1,0}(t)=u_t(t,x_0),\ b_{1,1}(t)=u_{tx}(t,x_0),\ b_{1,2}(t,x)=u_{txx}(t,x).$$

It is known that for any function $u\in W_{p,n}$ there exists a unique quadruple $b=(b_{1,2},\varphi,b_{1,0},b_{1,1})\in E_{p,n}$ by which we can represent this function in the form (7).

We can show that the opposite one is also valid. In other words, for any given quadruple $b=(b_{1,2},\varphi,b_{1,0},b_{1,1})$ from $E_{p,n}$ the function $u(x)$ determined by equality (6) belongs to the space $W_{p,n}$ and has the traces $u(t_0,x)$, $u_t(t,x_0)$, $u_{tx}(t,x_0)$ and dominating derivative $u_{txx}(t,x)$.

Now we try to choose the element $b=(b_{1,2},\varphi,b_{1,0},b_{11})$ so that corresponding function (7) satisfies conditions (2) and (3). For this we put function (7) in conditions (2) and (3). Then from (2) we obtain that

$$(l_0u)(x)=\varphi(x). \tag{8}$$

We obtain the following conditions from conditions (3)

$$(l_1u)(t)=b_{1,0}(t)\,\alpha_{1,1}+b_{1,1}(t)\,\alpha_{1,2}+$$
$$+\left[b_{1,0}(t)+\Delta b_{1,1}(t)+\int_{x_0}^{x_1}(x_1-\xi)\,b_{1,2}(t,\xi)\,d\xi\right]\beta_{1,1}+$$
$$+\left[b_{1,1}(t)+\int_{x_0}^{x_1}b_{1,2}(t,\xi)\,d\xi\right]\beta_{1,2},$$

$$
\begin{gathered}
(l_2u)(t) = b_{1,0}(t)\,\alpha_{2,1} + b_{1,1}(t)\,\alpha_{2,2}+ \\
+\left[b_{1,0}(t) + \Delta b_{1,1}(t) + \int\limits_{x_0}^{x_1} (x_1-\xi)\, b_{1,2}(t,\xi)\, d\xi \right] \beta_{2,1}+ \\
+\left[b_{1,1}(t) + \int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\, d\xi \right] \beta_{2,2},
\end{gathered} \tag{9}
$$

where $\Delta = x_1 - x_0$.

Denote by

$$
\begin{aligned}
\gamma_{1,1} &= \alpha_{1,1} + \beta_{1,1}, & \gamma_{1,2} = \alpha_{1,2} + \Delta\beta_{1,1} + \beta_{1,2}, \\
\gamma_{2,1} &= \alpha_{2,1} + \beta_{2,1}, & \gamma_{2,2} = \alpha_{2,2} + \Delta\beta_{2,1} + \beta_{2,2},
\end{aligned}
$$

$$
\begin{aligned}
\alpha_1(x) &= \beta_{1,1}(x_1 - x) + \beta_{1,2}, \\
\alpha_2(x) &= \beta_{2,1}(x_1 - x) + \beta_{2,2},
\end{aligned}
$$

and we rewrite equality (9) in the following form

$$
\begin{aligned}
b_{1,0}\gamma_{1,1} + b_{1,1}\gamma_{1,2} &= (l_1u)(t) - \int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\, a_1(\xi)\, d\xi, \\
b_{1,0}\gamma_{2,1} + b_{2,2}\gamma_{2,2} &= (l_2u)(t) - \int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\, a_2(\xi)\, d\xi.
\end{aligned} \tag{10}
$$

We can consider equality (10) as a system of linear algebraic equations with respect to $b_{1,0}(t)$ and $b_{1,1}(t)$.

With the help of $2n$ -dimensional matrix

$$
\gamma = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix}
$$

we write system (10) in the form of vector equation

$$
\begin{gathered}
(b_{1,0}(t), b_{1,1}(t))\,\gamma = \\
= \left((l_1u)(t) - \int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\, a_1(\xi)\, d\xi, \quad (l_2u)(t) - \int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\, a_2(\xi)\, d\xi \right).
\end{gathered} \tag{10$'$}
$$

Let the matrix γ be reversible and $\gamma^{-1} = \begin{pmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{pmatrix}$, where each of $k_{i,j}$ is some n-dimensional square constant matrix. Then from $(10')$ we obtain

$$
\begin{gathered}
(b_{1,0}(t), b_{1,1}(t)) = \\
= \left((l_1u)(t) - \int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\, a_1(\xi)\, d\xi, \quad (l_2u)(t) - \int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\, a_2(\xi)\, d\xi \right) \gamma^{-1}.
\end{gathered}
$$

Therefore

$$b_{1,0}(t)=\left[(l_1u)(t)-\int_{x_0}^{x_1}b_{1,2}(t,\xi)\,a_1(\xi)\,d\xi\right]k_{1,1}+$$
$$+\left[(l_2u)(t)-\int_{x_0}^{x_1}b_{1,2}(t,\xi)\,a_2(\xi)\,d\xi\right]k_{2,1},$$
$$b_{1,1}(t)=\left[(l_1u)(t)-\int_{x_0}^{x_1}b_{1,2}(t,\xi)\,a_1(\xi)\,d\xi\right]k_{1,2}+$$
$$=\left[(l_2u)(t)-\int_{x_0}^{x_1}b_{1,2}(t,\xi)\,a_2(\xi)\,d\xi\right]k_{2,2}. \tag{11}$$

Now taking into account (8) and (11) in equality (7) we obtain

$$u(t,x)=(l_0u)(x)+\int_{t_0}^{t_1}(l_1u)(\tau)\left[k_{1,1}+k_{1,2}(x-x_0)\right]d\tau+$$
$$+\int_{t_0}^{t_1}(l_2u)(\tau)\left[k_{2,1}+k_{2,2}(x-x_0)\right]d\tau+ \tag{12}$$
$$+\iint_G b_{1,2}(\tau,\xi)\left\{E(x-\xi)\,\theta(t-\tau)\,\theta(x-\xi)-\theta(t-\tau)\left[a_1(\xi)\,k_{1,1}+\right.\right.$$
$$\left.\left.+a_2(\xi)\,k_{2,1}+a_1(\xi)\,k_{1,2}(x-x_0)+a_2(\xi)(x-x_0)\right]\right\}d\tau d\xi$$

where $\theta(z)$ is a Heaviside function on the space R of real numbers, and E is $n\times n$ dimensional unit matrix.

Let

$$\beta_1(x)=k_{1,1}+k_{1,2}(x-x_0)$$
$$\beta_2(x)=k_{2,1}+k_{2,2}(x-x_0) \tag{13}$$
$$R_0(\tau,\xi;t,x)=\theta(t-\tau)\left[(x-\xi)\,\theta(x-\xi)\,E-a_1(\xi)\,\beta_1(x)-a_2(\xi)\,\beta_2(x)\right].$$

Now we write formula (13) in the following from

$$u(t,x)=(l_0u)(x)+\int_{t_0}^{t_1}(l_1u)(\tau)\,\beta_1(x)\,d\tau+$$
$$+\int_{t_0}^{t_1}(l_2u)(\tau)\,\beta_2(x)\,d\tau+\iint_G u_{txx}(\tau,\xi)\,R_0(\tau,\xi;t,x)\,d\tau d\xi. \tag{14}$$

Thus the following theorem is proved.

Theorem 1. *If* $\det \gamma \neq 0$, *then any function* $u \in W_{p,n}$ *is represented in the form of (14).*

It is obvious that the right hand side of formula (14) is defined by the values $(l_0 u)(x)$, $(l_1 u)(t)$, $(l_2 u)(t)$ and $u_{txx}(t,x)$ of the operators l_0, l_1, l_2 and $D_t D_x^2$ on the considered function $u(t,x)$. Therefore, we call the operators l_0, l_1, l_2 and $D_t D_x^2$ determining ones for representation (14).

Equivalent integral equation.

Beginning with formula (14) we seek the solution of problem (1), (2), (3) in the following form

$$u(t,x) = z_0(x) + \left[\int_{t_0}^{t_1} z_1(\tau)\beta_1(x) + z_2(\tau)\beta_2(x)\right] d\tau +$$

$$+ \iint_G b_{1,2}(\tau,\xi) R_0(\tau,\xi;t,x)\, d\tau d\xi, \tag{15}$$

where $b_{1,2} \in L_{p,n}(G)$ is an unknown function.

The fulfilment of initial condition (2) and nonlocal boundary conditions (3) for function (15) is obvious. Therefore, it remains to require from function (15) it were a solution of equation (1). For this we write function (15) in the form:

$$u(t,x) = g_0(t,x) + \overline{u}(t,x), \tag{16}$$

where

$$\begin{aligned} g_0(t,x) &= \varphi(x) + \int_{t_0}^{t_1} [z_1(\tau)\beta_1(x) + z_2(\tau)\beta_2(x)]\, d\tau, \\ \overline{u}(t,x) &= \iint_G b_{1,2}(\tau,\xi) R_0(\tau,\xi;t,x)\, d\tau d\xi. \end{aligned} \tag{17}$$

Then for function (15) we write equation (1) in the following form

$$(V_0 \overline{u})(t,x) = z(t,x), \tag{18}$$

where $z(t,x) = z_{1,2}(t,x) - (V_0 g_0)(t,x)$.

Expression (13) of the function $R_0(\tau,\xi;t,x)$ shows that the function $\overline{u}(t,x)$ has the following form

$$u(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} b_{1,2}(\tau,\xi)(x-\xi)\, d\tau d\xi -$$

$$-\int_{t_0}^{t}\int_{x_0}^{x} b_{1,2}(\tau,\xi)[a_1(\xi)\beta_1(x) + a_2(\xi)\beta_2(x)]\, d\tau d\xi.$$

Therefore the equalities

$$\overline{u}_t(t,x)=\int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\left[(x-\xi)\,\theta(x-\xi)\,E-a_1(\xi)\,\beta_1(x)-a_2(\xi)\,\beta_2(x)\right]d\xi,$$
$$\overline{u}_x(t,x)=\iint\limits_G b_{1,2}(t,\xi)\,\theta(x-\xi)\left[\theta(x-\xi)\,E-a_1(\xi)\,\beta_1'(x)-a_2(\xi)\,\beta_2'(x)\right]d\tau d\xi,$$
$$\overline{u}_{tx}(t,x)=\int\limits_{x_0}^{x} b_{1,2}(t,\xi)\left[\theta(x-\xi)\,E-a_1(\xi)\,\beta\prime(x)-a_2(\xi)\,\beta_2'(x)\right]d\xi, \tag{19}$$
$$\overline{u}_{xx}(t,x)=\int\limits_{t_0}^{t_1} b_{1,2}(\tau,\xi)\,\theta(t-\tau)\,d\tau,\quad \overline{u}_{txx}(t,x)=b_{1,2}(t,x)$$

are valid.

Using (19) from (18) we obtain

$$(V_0\overline{u})(t,x)=(Nb_{1,2})(t,x)\equiv b_{1,2}(t,\xi)+$$
$$+\iint\limits_G b_{1,2}(\tau,\xi)\,K_0(\tau,\xi;t,x)\,d\tau d\xi+\int\limits_{x_0}^{x_1} b_{1,2}(t,\xi)\,K_{0,1}(\xi;t,x)\,d\xi+ \tag{20}$$
$$+\int\limits_{t_0}^{t_1} b_{1,2}(\tau,\xi)\,K_{1,0}(\tau;t,x)\,d\tau=z(t,x)\,,\ (t,x)\in G,$$

where

$$K_0(\tau,\xi;t,x)=R_0(\tau,\xi;t,x)\,A_{0,0}(t,x)+$$
$$+\theta(t-\tau)\left[\theta(x-\xi)\,E-a_1(\xi)\,\beta_1\prime(x)-a_2(\xi)\,\beta_2'(x)\right]A_{0,1}(t,x)\,;$$
$$K_{1,0}(\tau;t,x)=\theta(t-\tau)\,A_{0,2}(t,x)\,;\ K_{0,1}(\xi;t,x)= \tag{21}$$
$$=\left[(x-\xi)\,\theta(x-\xi)\,E-a_1(\xi)\,\beta_1(x)-a_2(\xi)\,\beta_2(x)\right]A_{1,0}(t,x)+$$
$$+\left[\theta(x-\xi)\,E-a_1(\xi)\,\beta_1\prime(x)-a_2(\xi)\,\beta_2'(x)\right]A_{1,1}(t,x)\,.$$

Thus for defining $b_{1,2}(t,x)$ we obtain two-dimensional integral equation (20). Thus the following theorem is proved.

Theorem 2. *Let* $\det\gamma\neq 0$. *In order that problem (1), (2), (3) be always correctly solvable it is necessary and sufficient that integral equation (20) for any* $z\in L_{p,n}(G)$ *have a unique solution* $b_{1,2}\in L_{p,n}(G)$.

Existence and uniqueness of solution.

Let $b_{1,2}\in L_{p,n}(G)$ be a solution of equation (20). We write equation (20) in the following form

$$(N_1+N_2+I)\,b_{1,2}=z, \tag{22}$$

where I is a unit operator in the space $L_{p,n}(G)$ and

$$(N_1 b_{1,2})(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} b_{1,2}(\tau,\xi)\left[(x-\xi)A_{0,0}(t,x) + A_{0,0}(t,x)\right] d\tau d\xi +$$

$$+\int_{x_0}^{x} b_{1,2}(\tau,\xi)\left[(x-\xi)A_{1,0}(t,x) + A_{1,1}(t,x)\right] d\xi + \int_{t_0}^{t} b_{1,2}(\tau,\xi)A_{0,2}(t,x);$$

$$(N_2 b_{1,2})(t,x) = -\int_{t_0}^{t}\int_{x_0}^{x_1} b_{1,2}(\tau,\xi)\left\{\left[a_1(\xi)\beta_1(x) + a_2(\xi)\beta_2(x)\right]A_{0,0}(t,x) + \right. \tag{23}$$

$$\left. + \left[a_1(\xi)\beta_1'(x) + a_2(\xi)\beta_2'(x)\right]A_{0,1}(t,x)\right\} d\tau d\xi -$$

$$-\int_{x_0}^{x_1} b_{1,2}(\tau,\xi)\left\{\left[a_1(\xi)\beta_1(x) + a_2(\xi)\beta_2(x)\right]A_{1,0}(t,x) + \right.$$

$$\left. + \left[a_1(\xi)\beta_1'(x) + a_2(\xi)\beta_2'(x)\right]A_{1,1}(t,x)\right\} d\tau d\xi.$$

The operator N_1 acting n the space $L_{p,n}(G)$ is linear bounded and is a two-dimensional integral Volterra operator with respect to the point (t_0, x_0). Then the operator $(I + N_1)$ has a bounded inverse operator $B = (I + N_1)^{-1}$ acting in the space $L_{p,n}(G)$. Then from (22) we obtain that

$$b_{1,2} + BN_2 b_{1,2} = Bz. \tag{24}$$

Thus, the solutions of equations (22) and (24) in the space $L_{p,n}(G)$ are equivalent.

Now we estimate the norm $\|N_2\|$ of the operator $N_2 : L_{p,n}(G) \to L_{p,n}(G)$.

It is obvious that

$$\|(N_1 b_{1,2})(t,x)\|_n \le \int_{t_0}^{t}\int_{x_0}^{x} \|b_{1,2}(\tau,\xi)\|_n \left[\|\varphi_1(\xi,x)\|_{n\times n}\|A_{0,0}(t,x)\|_{n\times n} + \right.$$

$$+\left\|\frac{\partial\varphi_1(\xi,x)}{\partial x}\right\|_{n\times n}\|A_{0,1}(t,x)\|_{n\times n} d\tau d\xi + \int_{x_0}^{x}\|b_{1,2}(t,\tau)\|_n \times \tag{25}$$

$$\times\left[\|\varphi_1(\xi,x)\|_{n\times n}\|A_{1,0}(t,x)\|_{n\times n} + \left\|\frac{\partial\varphi_1(\xi,x)}{\partial x}\right\|_{n\times n}\|A_{1,1}(t,x)\|_{n\times n}\right],$$

where $\|\cdot\|_n$ and $\|\cdot\|_{n\times n}$ are Euclidean norms for n dimensional vectors and $n\times n$ dimensional matrices, respectively, and

$$\varphi_1(\xi,x) = a_1(\xi)\beta_1(x) + a_2(\xi)\beta_2(x).$$

Let

$$\varphi_1^0(\xi,x)=\|\varphi_1(\xi,x)\|_{n\times n},\quad a_i^0(\xi)=\|a_i(\xi)\|_{n\times n},$$
$$\left(\frac{\partial\varphi_1(\xi,x)}{\partial x}\right)^0=\left\|\frac{\partial\varphi_1(\xi,x)}{\partial x}\right\|_{n\times n},\quad \beta_i^0(\xi)=\|\beta_i(\xi)\|_{n\times n}.$$

Then we have

$$\varphi_1^0(\xi,x)\le a_1^0(\xi)\,\beta_1^0(x)+a_2^0(\xi)\,\beta_2^0(x). \tag{26}$$

Besides, from (13) we have

$$\beta_1'(x)=k_{1,2},\;\; \beta_2'(x)=k_{2,2}.$$

Therefore, the matrix $\frac{\partial\varphi_1(\xi,x)}{\partial x}$ is constant with respect to $x\in(x_0,x_1)$ and

$$\left(\frac{\partial\varphi_1(\xi,x)}{\partial x}\right)^0\le a_1^0(\xi)\,k_{1,2}^0+a_2^0(\xi)\,k_{2,2}^0, \tag{27}$$

where $k_{i,2}^0=\|k_{i,2}\|_{n\times n}$. Further form the expressions of the matrix functions $a_1(\xi)$ and $a_2(\xi)$ it follows that

$$\begin{aligned}a_1^0(\xi)&\le\beta_{1,1}^0(x_1-\xi)+\beta_{1,2}^0,\\ a_2^0(\xi)&\le\beta_{2,1}^0(x_1-\xi)+\beta_{2,2}^0,\end{aligned} \tag{28}$$

where $\beta_{i,j}^0=\|\beta_{i,j}\|_{n\times n}$.

Thus, using the Minkowsky inequality, from (26) and (27) we have

$$\|\varphi_1^0(\cdot,\ x)\|_{L_q(x_0,x_1)}\le\|a_1^0\|_{L_q(x_0,x_1)}\,q_1^0(x)+\|a_2^0\|_{L_q(x_0,x_1)}\,q_2^0(x) \tag{29}$$

and

$$\left\|\left(\frac{\partial\varphi_1^0(\cdot,\ x)}{\partial x}\right)^0\right\|_{L_q(x_0,x_1)}\le\|a_1^0\|_{L_q(x_0,x_1)}\,k_{1,2}^0(x)+\|a_2^0\|_{L_q(x_0,x_1)}\,k_{2,2}^0(x), \tag{30}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Further by virtue of equality (28) we have

$$\begin{aligned}\|a_1^0\|_{L_q(x_0,x_1)}&\le\beta_{1,1}^0\left(\frac{\Delta^{q+1}}{q+1}\right)^{\frac{1}{q}}+\beta_{1,2}^0\Delta^{\frac{1}{q}},\\ \|a_2^0\|_{L_q(x_0,x_1)}&\le\beta_{2,1}^0\left(\frac{\Delta^{q+1}}{q+1}\right)^{\frac{1}{q}}+\beta_{2,2}^0\Delta^{\frac{1}{q}},\end{aligned} \tag{31}$$

where $\Delta=x_1-x_0$. Therefore from inequalities (29) and (30) we obtain

$$\begin{aligned}\|\varphi_1^0(\cdot,\ x)\|_{L_q(x_0,x_1)}\le&\left[\beta_{1,1}^0\left(\frac{\Delta^{q+1}}{q+1}\right)^{\frac{1}{q}}+\beta_{1,2}^0\Delta^{\frac{1}{q}}\right]\left(k_{1,1}^0+\Delta k_{1,2}^0\right)+\\ &+\left[\beta_{2,1}^0\left(\frac{\Delta^{q+1}}{q+1}\right)^{\frac{1}{q}}+\beta_{2,2}^0\Delta^{\frac{1}{q}}\right]\left(k_{2,1}^0+\Delta k_{2,2}^0\right)\equiv\Delta_1\end{aligned} \tag{32}$$

and

$$\left\| \left(\frac{\partial \varphi_1^0 (\cdot,\ x)}{\partial x} \right)^0 \right\|_{L_q(x_0,x_1)} \leq \left[\beta_{1,1}^0 \left(\frac{\Delta^{q+1}}{q+1} \right)^{\frac{1}{q}} + \beta_{1,2}^0 \Delta^{\frac{1}{q}} \right] \Delta k_{1,2}^0 + \\ + \left[\beta_{2,1}^0 \left(\frac{\Delta^{q+1}}{q+1} \right)^{\frac{1}{q}} + \beta_{2,2}^0 \Delta^{\frac{1}{q}} \right] k_{2,2}^0 \equiv \Delta_2. \tag{33}$$

Now using Holder inequality from (25) we obtain

$$\left\| (N_2 b_{1,2}) (t,x) \right\|_n \leq \|b_{1,2}\|_{L_{p,n}(G)} (t_1 - t_0)^{\frac{1}{q}} \left[\left\| \varphi_1^0 (\cdot, x) \right\|_{L_q(x_0,x_1)} \|A_{0,0} (t,x)\|_{n\times n} + \right. \\ + \left\| \left(\frac{\partial \varphi_1 (\cdot, x)}{\partial x} \right)^0 \right\|_{L_q(x_0,x_1)} \|A_{0,1} (t,x)\|_{n\times n} + \|b_{1,2} (t,\cdot)\|_{L_{p,n}(x_0,x_1)} \times \\ \times \left[\|\varphi_1 (\cdot, x)\|_{L_q(x_0,x_1)} A_{1,0}^0 (x) + \left\| \left(\frac{\partial \varphi_1 (\cdot, x)}{\partial x} \right)^0 \right\|_{L_q(x_0,x_1)} A_{1,1}^0 (x) \right]. \tag{34}$$

Allowing for (32) and (33) in inequality (34) we have

$$\left\| (N_2 b_{1,2}) (t,x) \right\|_n \leq \\ \leq \|b_{1,2}\|_{L_{p,n}(G)} (t_1 - t_0)^{\frac{1}{q}} \left[\Delta_1 \|A_{0,0} (t,x)\|_{n\times n} + \Delta_2 \|A_{0,1} (t,x)\|_{n\times n} \right] + \\ + \|b_{1,2} (t,\cdot)\|_{L_{p,n}(x_0,x_1)} \left[\Delta_1 A_{1,0}^0 (x) + \Delta_2 A_{1,1} (x) \right].$$

From these inequalities by virtue of Minkowsky inequality we have

$$\|(N b_{1,2})\|_{L_{p,n}(G)} \leq \gamma_0 \|b_{1,2}\|_{L_{p,n}(G)},\ \forall b \in L_{p,n} (G), \tag{35}$$

where

$$\gamma_0 = (t_1 - t_0)^{\frac{1}{q}} \left[\Delta_1 \|A_{0,0}\|_{L_{p,n\times n}(G)} + \right. \\ \left. + \Delta_2 \|A_{0,1} (t,x)\|_{L_{p,n\times n}(G)} \right] + \Delta_1 \|A_{1,0}\|_{L_q(x_0,x_1)}.$$

From apriori estimation (35) obtained for the operator N we get

$$\|N_2\| \leq \gamma_0.$$

Then for the operator BN_2 we have

$$\|BN b_{1,2}\|_{L_{p,n}(G)} \leq \gamma_0 \|B\| \|b_{1,2}\|_{L_{p,n}(G)}. \tag{36}$$

Thus, if

$$\gamma_1 = \|B\| \gamma_0 < 1,$$

then equation (24) for any $z \in L_{p,n} (G)$ will have a unique solution $b_{1,2} \in L_{p,n} (G)$. It is obvious that at this solution $b_{1,2} \in L_{p,n} (G)$ equation satisfies the following condition, too

$$\|b_{1,2}\|_{L_{p,n}(G)} \leq \frac{1}{1-\gamma_1} \|B\| \|z\|_{L_{p,n}(G)}. \tag{37}$$

Thus the following theorem is valid.

Theorem 3. *If* $\det\gamma \neq 0$ *and* $\gamma_1 < 1$, *equation (20) for any* $z \in L_{p,n}(G)$ *has a unique solution* $b_{1,2} \in L_{p,n}(G)$ *and for this solution estimation (37) is valid.*

Theorems 2 and 3 show that the following theorem is also valid

Theorem 4. *If* $\det\gamma \neq 0$ *and* $\gamma_1 < 1$, *problem (1), (2), (3) is everywhere correctly solvable.*

Remark. Expressions (32) and (33) of the numbers Δ_1 and Δ_2 show that if

$$\beta_{i,j} = 0, \quad i,j = 1,2 \tag{38}$$

then $\Delta_1 = \Delta_2 = 0$. Therefore by virtue of (35) we have $\gamma_0 = 0$. Consequently, in this case $\gamma_1 = 0$. Thus in this case problem (1), (2), (3) is everywhere correctly solvable if

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \neq 0. \tag{39}$$

Note that if conditions (38) and (39) are fulfilled, we can consider boundary condition (3) together with initial conditions (2) as a new type Goursat condition. In particular, when $\alpha_{1,1} = \alpha_{2,2} = E$, $\alpha_{1,2} = \alpha_{2,1} = 0$ conditions (13) have the form

$$\begin{aligned} u_t(t, x_0) &= z_1(t), \\ u_{tx}(t, x_0) &= z_2(t), \end{aligned} \tag{40}$$

where $z_i(t) \in L_{p,n}(t_0, t_1)$. The paper [5] is devoted to the solution of problems (1), (2) and (40).

References

[1]. Sobolev S.L. *Some application of functional analysis in mathematical physics.* Novosibirsk. 1962. (Russian)

[2]. Shkhanukov M.Kh. *On some boundary value problems for the third order equations arising at modeling filtration of fluid in porous medium.* Differen. uravnenija, 1982, v.18, No4, pp.689-699. (Russian)

[3]. Shkhanukov M.Kh. *On some boundary value problems for the third order equation and extremal properties of its solutions.* Differen. uravnenija, 1983, v.19, No1, pp.145-152. (Russian)

[4]. Akhiev S.S. *On general form of linear bounded functionals in one functional S.L.Sobolev space.* Dokl. AN Azerb. SSR, 1979, v.35, No6, pp.3-7. (Russian)

[5]. Yusifov M.R. *Riemann function for the third order pseudoparabolic equations.* Transaction of NAS of Azerbaijan, 2003, v.XXIII, No4, pp.193-197.

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