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ON INTERNAL AND EXTERNAL HEMMING INDICES OF k-VALUED CLONES II

Abstract

The proofs of several theorems, announced in the first part of the paper are given, and, moreover external Hemmong's index of stabilizer of binary central ratio with one block of the pairs has been calculated.

The paper continues the author's work [1], in which at arbitrary $k \geq 3$ research of behaviour on the lattice $S(\mathcal{F})$ of all subclones $C \leq \mathcal{F}_k$ of k-valued logic $\mathcal{F}_k$ of two natural numerical characteristics of internal Hemming's index $H_k(C) := \sup\limits_{n\in N} h_k(n, C)$ and external Hemming's index $H'_k(C) := \sup\limits_{n\in N} h'_k(n, C)$ is started, where

$$h_k(n, C) := \max\left\{ \rho_n\left(f, F_k^{(n)} \backslash C^{(n)}\right) \middle| f \in C^{(n)} \right\},$$

$$h'_k(n, C) := \max\left\{ \rho_n\left(g, C^{(n)}\right) \middle| g \in F_k^{(n)} \backslash C^{(n)} \right\}, \ \rho_n(f, g) := d_n(f, g) / k^n$$

"fractional version" of distance of Hemming $d_n(f, g)$ between functions $f, g \in F_k^{(n)}$ and for set $A \subseteq F_k$ $A^{(n)}$ denotes subset of all n-arie $f \in A$.

In [1] it is being explained, why at k=2the calculation of mentioned indices turned out to be relatively easy problem – due to obtained by Post the clear description of lattice $S(\mathcal{F}_2)$ and following from it absence of infinite strictly increasing chains in it; at $k = 2$ the internal indices $H_2(C)$ under the other name was calculated in [3], and external indices $H'_2(C)$ were also easy calculated. In [1] it is also indicated, that by virtue of established in [4] continual lattice $S(\mathcal{F}_k)$ at $k \geq 3$ and by availability of infinite strictly increasing chains in it, the principles of the work [3] and its modifications inapplicable for calculation of indices $H_k(C)$, $H'_k(C)$ at $k \geq 3$, and it is mentioned fulfilled calculation $H'_3(C)$ in [5] for maximum subclones $C < \mathcal{F}_3$ and some their intersections (the internal Hemming's indices in [5] aren't considered at all).

In [1] at arbitrary $k \geq 3$ we calculated $H'_k(C)$ for some series of subclones $C < \mathcal{F}_k$ (see in [1] conjecture 1-3 and theorem 1) and in view of restrictions on volume of the publications without proof we formulated theorems 2, 3 about external indices $H'_k(C)$ and theorem 4 about internal indices $H_k(C)$ indicating, that their proof will be in the second part of the paper. Now we shall give these proofs, and then we'll formulate and prove theorem 5, which isn't given in [1].

Let's remind the formulations of theorems 2-4, at that for theorem 4 we'll give the extended, with respect to [1] formulation.

In the special notation we follow, in general, [1,6,9], but in common terminology and notation the works [7, 8, 9]. Here we'll remind the following denotation: for non-identical permutations π from symmetrical group σ_k of the set $E_k := \{0, 1, ..., k-1\}$ S_x denotes a clone of all functions $f \in F_k$, self-dual relative to π (in particular, at $\pi(x) := x \oplus a$, where $a \in E_k$ and $\oplus$ is adding by module k, we write $S_{x\oplus a}$); for equivalence θ_a on E_k with blocks $\{a\}$ and $E\backslash\{a\}$ $(a \in E_k)$ $U_a := St_k(\theta_a) := \{f \in F_k |\ f \text{ keeps } \theta_a\}$ (the common definition of stabilizer $St_k(\rho)$ of arbitrary ratio $\rho \subseteq E_k^m$ also is given in terms of preservation by functions $f \in F_k$ of ratio ρ).

Theorm 2. *For any $k \geq 3$ and any $\pi \in \sigma_k$, which is decomposed in the product of independent cycles of the same prime length p an inequality $H_k'(S_\pi) = p - 1/p$ is true. In particular, at the prime $k = p \geq 3$ (here expansion consists of single cycle, $\pi(x)$ coincides with one of cyclic permutations $x \oplus 1, ..., x \oplus (p-1)$ and $S_{x\oplus 1} = ... = S_{x\oplus(p-1)}$) also we have $H_p'(S_{x\oplus 1}) = \frac{p-1}{p}$.*

Proof. Let's start from particular case of prime $p \geq 3$ and clone $S_{x\oplus 1}$. Cyclic subgroup G, generated in σ_p by permutations $\Im(x) := x \oplus 1$, consists of $\Im, \Im^2, ..., \Im^{p-1}$, $id_{E_p} = e_1^1$ and $\Im^j(x) = x \oplus j$. Therefore, by any $n \in N$ the action of group G divides E_p^n to orbits of the form:

$$\{<\alpha_1, ..., \alpha_n>, <\alpha_1 \oplus 1, ..., \alpha_n \oplus 1>, ..., <\alpha_1 \oplus (p-1), ..., \alpha_n \oplus (p-1)>\},$$

$$<\alpha_1, ..., \alpha_n> \in E_p^n.$$

Each orbit consists of vectors p (at any n) and number of orbits equals $p^n/p = p^{n-1}$.

Arbitrary choosing in each orbit by one representative, we'll denote this system of representatives by R_G;then $|R_G| = p^{n-1}$.

It is obvious, that for any $f \in F_p^{(n)} \backslash S_{x\oplus 1}^{(n)}$ there exists a single $g_f \in S_{x\oplus 1}^{(n)}$, coinciding with f on all vectors from R_G: this g_f is uniquely defined $\Im$-self-dual extension of particular function $f \upharpoonright R_G$ (here $\upharpoonright$ is the restriction of function on a part of range of definition).

Further by construction

$$\rho_n(f, g_f) \leq \frac{p^n - |R_G|}{p^n} = \frac{p^n - p^{n-1}}{p^n} = \frac{p-1}{p}.$$

is fulfilled.

From this estimation and uniqueness of g_f if follows the estimation $\rho_n\left(f, S_{x\oplus 1}^{(n)}\right) \leq \frac{p-1}{p}$ from which in view of the arbitrariness of $f \in F_p^{(n)} \backslash S_{x\oplus 1}^{(n)}$ we conclude, that

$$\max\left\{\rho_n\left(f, S_{x\oplus 1}^{(n)}\right) \middle| f \in F_p^{(n)} \backslash S_{x\oplus 1}^{(n)}\right\} \leq \frac{p-1}{p}.$$

So, for any $n \in N$ $h_p'(n, S_{x\oplus 1}) \leq \frac{p-1}{p}$ and therefore $H_p'(S_{x\oplus 1}) \leq \frac{p-1}{p}$. On the other hand, it is easy to see, that $S_{x\oplus 1}^{(1)} = \mathbf{G} = \{e_1^1, \Im, ..., \Im^{p-1}\}$ and constant function c_0^1 coincides with e_1^1 only at the point 0, but with $\Im^j$ $(j \in \{1, ..., p-1\})$ only at the point $(\Im^j)^{-1}(0)$ from which $\rho_1(c_0^1, e_1^1) = \rho_1(c_0^1, \Im) = ... = \rho_1(c_0^1, \Im^{p-1}) = \frac{p-1}{p}$ and, consequently, $\rho_1\left(c_0^1, S_{x\oplus 1}^{(1)}\right) = \frac{p-1}{p}$. Since $c_0^1 \in F_p^{(1)} \backslash S_{x\oplus 1}^{(1)}$ we obtain $h_p'(1, S_{x\oplus 1}) \geq \rho_1\left(c_0^1, S_{x\oplus 1}^{(1)}\right) = \frac{p-1}{p}$ and consequently, $H_p'(S_{x\oplus 1}) \geq h_p'(1, S_{x\oplus 1}) \geq \frac{p-1}{p}$. So, $H_p'(S_{x\oplus 1}) = \frac{p-1}{p}$ for any prime $p \geq 3$.

Let now $k \geq 3$ arbitrary natural number and permutation $\pi \in \sigma_k$ is expanded into the product of independent cycles $\Im_1, ..., \Im_{k/p}$ of the same prime length p.

Without restriction of generality we can count, that cycle $\Im_1$ acts on the set $A_1 := \{0, ..., p-1\}, ...,$ cycle $\Im_{k/p}$ acts on the set $A_{k/p} := \{k-p-1, ..., k-1\}$. Cycles $\Im_j$ commute pairwise and therefore $\pi^2 = \left(\Im_1 \circ \cdots \circ \Im_{k/p}\right)^2 = \Im_1^2 \circ \cdots \circ \Im_{k/p}^2$, $\pi^3 = \Im_1^3 \circ \cdots \circ \Im_{k/p}^3$ and etc., moreover, permutation $\Im_j^p$ at any $j \in \left\{1, ..., \frac{k}{p}\right\}$ coincides with id_{A_j}. Consequently, cyclic subgroup G, generated in σ_k by permutation π consists of $e_1^1 id_{E_k},\ \pi^2 = \Im_1 \circ \cdots \circ \Im_{k/p},\ \pi^2 = \Im_1^2 \circ \cdots \circ \Im_{k/p}^2$.

By any $n \in N$ the action of a group G divides E_k^n to the orbits of type

$$\left\{< \alpha_1, ..., \alpha_n >, < \pi(\alpha_1), ..., \pi(\alpha_n) >, ..., < \pi^{p-1}(\alpha_1), ..., \pi^{p-1}(\alpha_n) >\right\} =$$

$$= \left\{< \alpha_1, ..., \alpha_n >, < \Im_{j_1}(\alpha_1), ..., \Im_{j_n}(\alpha_n) >, ..., < \Im_{j_1}^{p-1}(\alpha_1), ..., \Im_{j_n}^{p-1}(\alpha_n) >\right\},$$

where $\alpha_m \in A_{j_m}\left(m = \overline{1,n}\right)$; obviously, A_{j_m} is uniquely defined by α_m.

As in considered above particular case (prime $p \geq 3$ and clone $S_{x\oplus 1}$) we have: each orbit consists of vectors p (for any $n \in N$); number of orbits equals $\frac{k^n}{p}$ and, chosen in each orbit by representative, we get the system of representatives R_G, of capacity $|R_G| = \frac{k^n}{p}$; for any $f \in F_k^{(n)} \backslash S_\pi^{(n)}$ there exists a single $g_f \in S_\pi^{(n)}$ at that $g_f \upharpoonright R_G = f \upharpoonright R_G$; moreover $\rho_n(f, g_f) \leq \frac{k^n - \left(\frac{k^n}{p}\right)}{k^n} = \frac{k^n\left(1-\frac{1}{p}\right)}{k^n} = \frac{p-1}{p}$, whence in view of the uniqueness of g_f we get $\rho_n\left(f, S_\pi^{(n)}\right) \leq \frac{p-1}{p}$ and by virtue of arbitrariness of $f \in F_k^{(n)} \backslash S_\pi^{(n)}$ we come to the estimation $h_k'(n, S_\pi) \leq \frac{p-1}{p}$, and in view of arbitrariness $n \in N$ we obtain $H_k'(S_\pi) \leq \frac{p-1}{p}$.

For obtaining lower estimation $H_k'(S_\pi) \geq \frac{p-1}{p}$ (but for it, it's enough to set $h_k'(1, S_\pi) \leq \frac{p-1}{p}$) we'll consider a vector $\hat{0} :=< 0, p, ..., k-p-1 >$ composed from left ends of segments $A_1, ..., A_{k/p}$, and π-quasi constant unari function $qc_{\hat{0}}(x)$, given by equalities $qc_{\hat{0}}(A_1) = \{0\}, qc_{\hat{0}}(A_2) = \{p\}, ..., qc_{\hat{0}}\left(A_{k/p}\right) = \{k-p-1\}$ .It is easy to see, that $qc_{\hat{0}} \notin S_\pi$: for example, $\left(qc_{\hat{0}} \circ \pi\right)(0) = qc_{\hat{0}}(\pi(0)) = 0$ (here necessary membership for equality to zero $\pi(0) \in A_1$ is provided, by virtue of simplicity of p such that $p \geq 2$ is true), whereas $\left(\pi \circ qc_{\hat{0}}\right)(0) = \pi(0) \neq 0$. Moreover: $qc_{\hat{0}}$ coincides with $e_1^1 = id_{E_k}$ only at points $0, p, ..., k-p-1$ (in coordinates of vectors $\hat{0}$); coincides with π only at points $\pi^{-1}(0), \pi^1(p), ..., \pi^{-1}(k-p-1)$; coincides with π^2 only at points $\left(\pi^2\right)^{-1}(0), \left(\pi^2\right)^{-1}(p), ..., \left(\pi^2\right)^{-1}(k-p-1)$ and etc., i.e., coincides with permutations $\pi^j \in \mathbf{G} = \left\{e_1^1, \pi, ..., \pi^{p-1}\right\}$ exactly at p points (by one of each $A_m\,(m \in \{1, ..., k/p\}$); otherwise distinguishe exactly at $(p-1)\frac{k}{p}$ points.

Consequently

$$\rho_1\left(qc_{\hat{0}}, e_1^1\right) = \rho_1\left(qc_{\hat{0}}, \pi\right) = ... = \rho_1\left(qc_{\hat{0}}, \pi^{p-1}\right) = \frac{(p-1)\,k}{pk} = \frac{p-1}{p}.$$

On the other hand, as well known from the theory of groups $\sigma_k \cap S_\pi^{(1)}$ besides permutations from $\mathbf{G} = \left\{e_1^1, \pi, ..., \pi^{p-1}\right\}$ contains only compositions $\pi_1 \circ \pi_2$, where one of the sets is a product of any degrees of the cycles $\Im_j\,(j \in \{1, ..., k/p\})$ of permutation π, and another for any fixed $i \in \{1, ..., p\}$ permutations i-th elements of the segments $A_1, ..., A_{k/p}$. It is easy to check, that for such $\pi_1 \circ \pi_2$, $\rho_1\left(qc_{\hat{0}}, \pi_1 \circ \pi_2\right) = \frac{p-1}{p}$ is also true, and $\rho_1\left(qc_{\hat{0}}, \varphi\right) \geq \frac{p-1}{p}$ for $\varphi \in S_\pi^{(1)} \backslash \sigma_l$. Consequently,

$\rho_1\left(qc_{\hat{0}}, S_\pi^{(1)}\right) = \frac{p-1}{p}$ form which $h_k'(1, S_\pi) \geq \frac{p-1}{p}$ and $H_k'(S_\pi) \geq \frac{p-1}{p}$.

So, $H_k'(S_\pi) = \frac{p-1}{p}$ and the theorem is proved.

Theorem 3. *At any odd $k \geq 3$ for any $a \in E_k$ $H_k'(U_a) = \frac{1}{2}$.*

Proof. Let θ_a^n be an equivalence on E_k^n, inducted by equivalence θ_a, i.e., for any vectors $\tilde{L} =< \alpha_1, ..., \alpha_n >,\ \tilde{\beta} =< \beta_1, ..., \beta_n >\in E_k^n$ it is fulfilled

$$\tilde{\alpha} \sim \tilde{\beta}\,(\mathrm{mod}\,\theta_a^n) \iff (\forall i \in \{1, ..., n\})\,\alpha_i \sim \beta_i\,(\mathrm{mod}\,\theta_a)\,.$$

For the subset $J = \{j_1, ..., j_S\} \subseteq \{1, ..., n\}$ we shall suppose

$$B_J := \{< \alpha_1, .., \alpha_n >\in E_k^n\, |(\forall j \in \{1, ..., n\})\ (\alpha_j = a \iff j \in J)\}\,;$$

the case $J = \varnothing$ (i.e. $S = 0$) isn't being excepted. Then θ_a^n divides E_k^n into blocks $B_J\,(\varnothing \subseteq J \subseteq \{1, ..., n\})$, where $B_{\{j_1,...,j_S\}}$ consists of $(k-1)^{n-S}$ vectors, and number of blocks of capacity $(k-1)^{n-S}$ equals C_n^S.

In particular, a single block of capacity $(k-1)^n$ is $B_\varnothing = (E_k \backslash \{a\})^n$, and a single block of capacity 1 will be $B_{\{1,...,n\}} = \{< a, ..., a >\}\,.$

Arbitrary function $g \in U_a^{(n)}$ on each of the blocks B_J takes either only value a, or only value from $E_k \backslash \{a\}$.

For the function $\varphi \in F_k^{(1)}$ and the block B_J we'll suppose

$$B_J^a := \{\tilde{\alpha} \in B_J |\, \varphi\,(\tilde{\alpha}) = a\}\,,\quad B_J^{E_k\backslash\{a\}}\,(\varphi) := \{\,\tilde{\alpha} \in \mathcal{B}_J |\, \varphi\,(\tilde{\alpha}) \in E_k \backslash \{a\}\}$$

and by $\tilde{B}\,(\varphi)$ we'll denote unification of all these blocks B_J, for which one of the sets $B_J^a\,(\varphi)\,,\ B_J^{E_k\backslash\{a\}}\,(\varphi)$ is empty (i.e. these blocks, within of which φ keeps θ_a). Obviously, $\tilde{B}\,(\varphi)$ is always non-empty (i.e. for all φ), as $< a, ..., a >\in \tilde{B}\,(\varphi)\,;$ really, for block $B_{\{1,...,n\}} = \{< a, ..., a >\}$ one of the sets $B_{\{1,...,n\}}^a\,(\varphi)\,,\ B_{\{1,...,n\}}^{E_k\backslash\{a\}}\,(\varphi)$ is empty, since this block consists of the unique collection. On the other hand, for any $f \in F_k^{(n)} \backslash U_a^{(n)}, \tilde{B}\,(f) \neq E_k^n$, as at least on the one block $B_{J_0} \neq B_{\{1,...,n\}}$the function f takes both the value a, at least one value of $E_k \backslash \{a\}$, and therefore $B_{J_0} \nsubseteq \tilde{B}\,(f)$ whence $B_{J_0} \cap \tilde{B}\,(f) = \varnothing$.

Let's consider arbitrary $f \in F_k^{(n)} \backslash U_a^{(n)}$. By virtue of above-stated, there exist blocks of B_y with condition $B_y \cap \tilde{B}\,(f) = \varnothing$ and for each of them both sets $B_y^a\,(f)\,, B_y^{E_k\backslash\{a\}}\,(f)$ are non-empty. For every such block B_y we'll suppose

$$B_j^* := \left\{ \begin{array}{ll} B_J^a\,(f)\,,\ \text{if} & |B_J^a\,(f)| \leq \left|B_J^{E_k\backslash\{a\}}\,(f)\right|, \\ B_J^{E_k\backslash\{a\}}\,(f) & \text{otherwise} \quad , \end{array} \right.$$

$B_J^{**}\,(f) := B_J \backslash B_J^*\,(f)$, and we'll consider the sets $\dot{+}\left\{B_J^*\left(f\left|B_J \cap \widetilde{B}\,(f) = \varnothing\right.\right)\right\}$ (let's denote it by $B^*\,(f)$) and $\dot{+}\left\{B_J^{**}\left(f\left|B_J \cap \widetilde{B}\,(f) = \varnothing\right.\right)\right\}$ (let's denote it by $B^{**}\,(f)$); it is clear that $B^{**}\,(f) = E_k^n \backslash \left(\widetilde{B}\,(f) + B^*\,(f)\right)$. Then $E_k^n = \widetilde{B}\,(f) \dot{+}$ $\dot{+} B^*\,(f) \dot{+} B^{**}\,(f)$all of addends are non-empty (here and above it means a unification of pairwise nonintersecting nonempty sets) and $|B^*\,(f)| \leq \frac{1}{2}\left(k^n - \left|\widetilde{B}\,(f)\right|\right) \leq \frac{1}{2}\,(k^n - 1)$.

For vector $\tilde{x} \in E_k^n$ let's denote the lower index $J_{\tilde{x}}$ of that unique block B_J for which by $\tilde{x} \in B_J$. Let's consider the following function

$$g_f\,(\tilde{x}) := \left\{ \begin{array}{lll} f\,(\tilde{x})\,, & if & \tilde{x} \in \widetilde{B}\,(f) \dot{+} B^{**}\,(f) \\ f\left(\tilde{L}^\circ_{J_{\tilde{x}}}\right), & if & \tilde{x} \in B^*\,(f)\,. \end{array} \right.$$

arbitrariry fixing in every of the sets $B_J^{**}(f)$ by one vector $\tilde{\alpha}_J^0$.

It is easy to see, that $g_f \in U_a^{(n)}$ and $\rho_n(f, g_f) = \frac{|B^*(f)|}{k^n} \leq \frac{1}{2} \cdot \frac{k^n - 1}{k^n} < \frac{1}{2}$. Consequently, $\rho_n\left(f, U_a^{(n)}\right) < \frac{1}{2}$ and in the view of arbitrariness $f \in F_k^{(n)} \backslash U_a^{(n)}$ we come to $h'_k(n, U_a) < \frac{1}{2}$; here inequality is strict, since by finding $h'_k(n, U_a)$ the maximum is taken by finite set $F_k^{(n)} \backslash U_a^{(n)}$. Therefore $H'_k(U_a) = \sup\limits_{n \in N} h'_k(n, U_a) \leq \frac{1}{2}$.

We'll show, that strict inequality $H'_k(U_a) < \frac{1}{2}$ is impossible. Lets assume contrary. Then $(\forall n \in N)\, h'_k(n, U_a) < \frac{1}{2}$ and there exists sufficiently large $n_0 \in N$ such, that $h'_k(n_0, U_a) < \frac{1}{2} - \frac{1}{k^{n_0}}$, as in contrary case $H'_k(U_a) = \sup\limits_{n \in N} h'_k(n, U_a) \leq \frac{1}{2}$ in spite of assumption $H'_k(U_a) < \frac{1}{2}$.

Let's consider different from $\{< a, ..., a >\}$ blocks B_J of equivalence $\theta_a^{n_0}$ on $E_k^{n_0}$, i.e., index $J = \{j_1, ..., j_s\}$ should be taken different from $\{1, ..., n_0\}$ and therefore $s < n_0$. Oddness of the number $k \geq 3$ implies evenness of all numbers $(k-1)^{n_0 - s}$ at $s < n_0$. Consequently, all blocks B_J different from $B_{\{1,...,n_0\}} = \{< a, ..., a >\}$ consist of even number of vectors (we'll remind, that block $B_{\{j_1,...,j_s\}}$ consists of $(k-1)^{n-s}$ vectors). Therefore in $F_k^{(n_0)} \backslash U_a^{(n_0)}$ there exists a function f such, that $\left|\widetilde{B}(f)\right| = 1$ (i.e. $\widetilde{B}(f) = \{< a, ..., a >\}$) and $|B_J^*(f)| = |B_J^{**}(f)| = \frac{1}{2} \cdot |B_J|$ for any $J \neq \{1, ..., n_0\}$, from which $|B^*(f)| = |B^{**}(f)| = \frac{1}{2}(k^{n_0} - 1)$. It is easy to see, that $\rho_{n_0}\left(f, U_a^{(n_0)}\right) \geq \frac{1}{2} \frac{k^{n_0} - 1}{k^{n_0}} = \frac{1}{2} - \frac{1}{2k^{n_0}} > \frac{1}{2} - \frac{1}{k^{n_0}}$ and, consequently, $h'_k(n_0, U_a) > \frac{1}{2} - \frac{1}{k^{n_0}}$ is contradiction with earlier deduced inequality $h'_k(n_0, U_a) \leq \frac{1}{2} - \frac{1}{k^{n_0}}$ from supposition $H'_k(U_a) < \frac{1}{2}$. So, assumption $H'_k(U_a) < \frac{1}{2}$ is not true, i.e., $H'_k(U_a) = \frac{1}{2}$ and the theorem is proved.

Let $\widetilde{S}(\mathcal{F}_k)$ be a family of all proper sub clones of the clone $\mathcal{F}_k$ and $\widetilde{S}^{\max(f_k)}$ be a subfamily of all maximum elements of particularly ordered family $\widetilde{S}(\mathcal{F}_k) := \left(\widetilde{S}(\mathcal{F}_k); \subseteq\right)$. Let, further EQ_k^* be a set of all non-trivial equivalences on E_k and for $\theta \in EQ_k^*$ $U_\theta := ST_k(\theta)$.

Let's denote a family $\{\tau_E \,|\varnothing \subset E \subset E_k\} \cup \{U_\theta \,|\theta \in EQ_k^*\} \cup \{S_\pi \,|\pi \in \sigma_k \backslash \{e_1^1\}\}$,by $\mathcal{R}$, but its subfamily $\{\tau_E \,|\varnothing \subset E \subset E_k\} \cup \{U_\theta \,|\theta \in EQ_k^*\} \cup \{S_\pi \,|\pi \in \sigma_k$ which is being expanded to the product of independent cyclesof the same simple length$\}$ by $\mathcal{R}^*$.

As well known (see, for example, [6,7]), $\mathcal{R}^* \subset \widetilde{S}^{\max}(\mathcal{F}_k)$, but owing to the clones from $\mathcal{R} \backslash \mathcal{R}^*$ $\mathcal{R} \not\subseteq \widetilde{S}^{\max}(\mathcal{F}_k)$ and $S_\pi \in \widetilde{S}^{\max}(\mathcal{F}_k) \Longleftrightarrow S_\pi \in \mathcal{R}^*$.

Further (see [7]), the family of stabilizers of various bounded particular orders on E_k we'll denote by BO, by $Q\mathcal{L}$-the family (non-empty only at $k = p^m$, where p is prime and $m \geq 1$) of clones of G- quasilinear functions relative to various elementary Abelian p-groups G with support E_k, $Z^{(\geq 2)}$-be family of stabilizers of central ratios ρ of arity $a(\rho) \geq 2$ (and obviously $a(\rho) \leq k-1$), by SH-a family of stabilizers of strong homomorphism preimages of elementary q-adic ratios.

At $A \subseteq \widetilde{S}^{\max}(\mathcal{F}_k)$ we'll say, that $C < \mathcal{F}_k$ is A-local, if C isn't contained in $D \in \widetilde{S}^{\max}(\mathcal{F}_k) \backslash A$.

Theorem 4. *At any $k \geq 3$ $H_k(C) = 1$ for all $C < F_k$, from the chain of Burle (see [1, 10]) and $H_k(C) = 1/k$ for all other $C < F_k$, except at most countable set $\left(Z^{(\geq 2)} \cup SH\right)$ of localized subclones. In particular, $H_k(St_k(\rho)) = l/k$ for any l-ary $(2 \leq l \leq k-1)$ of the central ratio ρ and $H_k(St_k(\rho)) = q/k$ for any q-adic $(3 \leq q \leq k-1)$ ratio ρ is strong zoomorphic preimage of elementary q-ary ratio.*

Corollary. *There exist a clone C and its proper subclones $C_1, ..., C_m$ such, that*

$C = C_1 \cup ... \cup C_m,\ H_k(C_j) = 1/k$ *for all* $j \in \{1, ..., m\}$ *but* $H_k(C) > 1/k$. *Further* $H_k(C) = 1/k$ *for all minimum subclones* $C < F_k$.

In the proof of the theorem and corollary the following well-known facts are being used, which we'll formulate in the lemmas 1-4.

Lemma 1. ("folklore", see, for example [6, 7, 8]). *The clone* $\mathcal{F}_k$ *is 1-generated, every* $C < \mathcal{F}_k$ *is contained in some*

$$D \in \widetilde{S}^{\max}(\mathcal{F}_k) \text{ and } \cup \{C | C < \mathcal{F}_k\} = \cup \left\{ D | D \in \widetilde{S}^{\max}(\mathcal{F}_k) \right\},$$

but family $\widetilde{S}^{\max}(\mathcal{F}_k)$ *is finite.*

Lemma 2. (see, for example,[6, 7]). *At any* $n \geq 2$ $F_k^{(n)}$ *generates* $\mathcal{F}_k$ *and therefore for any* $C < \mathcal{F}_k$ $(\forall n \geq 2)$ $C^{(n)} \neq F_k^{(n)}$ *is true.*

Lemma 3. (Russo's theorem[11]). *At* $k \geq 3$ *and* $n \geq 2$ *the function* $f \in F_k^{(n)}$ *generates* $\mathcal{F}_k$ *if and only if* $(\forall C \in \mathcal{R})\, f \notin C$,*, that is equivalent to the condition* $(\forall C \in \mathcal{R}^*)\, f \notin C$*; otherwise* $\cup \{F | \mathcal{F} \in \mathcal{R}^*\} = \cup \left\{ D | D \in \widetilde{S}^{\max}(\mathcal{F}_k) \right\}$.

Lemma 4. (Rozenberque's theorem [7]).

$$\widetilde{S}^{\max}(\mathcal{F}_k) = \mathcal{R}^* \cup BO \cup Q\mathcal{L} \cup \mathcal{Z}^{(\geq 2)} \cup SH.$$

Let's return to the proof of the theorem. Equality $H_k(C) = 1$ for all $C < \mathcal{F}_k$ from the chain of Burle is explained already at the remark 3 of [1], and the equality $H_k(Sel_k) = \frac{1}{k}$ was verified at the proof of proposition 1 from [1]. By virtue of this equality, monotonicity of the mapping H_k of lemma 1 and because of the unique representative of Burle's chain in family $\widetilde{S}^{\max}(\mathcal{F}_k)$ is a clone of Slupetsky τ_{Sl}, it is enough to prove $H_k(C) = \frac{1}{k}$ only for all members of family $\widetilde{S}^{\max}(\mathcal{F}_k) \setminus \{\tau_{Sl}\}$. Let $C \in \widetilde{S}^{\max}(\mathcal{F}_k) \setminus \{\tau_{Sl}\}$ and therefore $C^{(1)} \neq F_k^{(1)}$. Then for any $n \in N$ and any $f \in C^{(n)}$ $\rho_n\left(f, F_k^{(n)} \backslash C^{(n)}\right) \geq 1/k^n$ from which by virtue of arbitrariness $f \in C^{(n)}$ $h_k(n, C) = \max\left\{ \rho_n\left(f, F_k^{(n)} \backslash C^{(n)}\right) \middle| f \in C^{(n)} \right\} \geq \frac{1}{k^n}$ and consequently,

$$H_k(C) = \sup_{n \in N} h_k(n, C) \geq h_k(1, C) \geq \frac{1}{k}.$$

On the other hand it is easy to see, that for any $C \in \mathcal{R}^*$ $H_k(C) \leq \frac{1}{k}$ whence taking into account $(\forall C \in \mathcal{R}^*)\, C \neq \tau_{Sl}$ and established above inequality $H_k(C) \geq \frac{1}{k}$ for all $C \in \widetilde{S}^{\max}(\mathcal{F}_k) \setminus \{\tau_{Sl}\}$ we come to $(\forall C \in \mathcal{R}^*)$ $H_k(C) = \frac{1}{k}$.

Really, for $C \in \mathcal{R}^*$ for any $n \in N,\ f \in C^{(n)}$ it is fulfilled $\rho_n\left(f, F_k^{(n)} \backslash C^{(n)}\right) \leq \frac{1}{k^n}$, as for f exists $g \in F_k^{(n)} \backslash C^{(n)}$, distinguished from f only for one vector from E_k^n (it follows from specific character of clones $\tau_E, U_\oplus, S_\pi$), i.e., $\rho_n(f, g) = \frac{1}{k^n}$ but taking of minimum by $g \in F_k^{(n)} \backslash C^{(n)}$ reduces to the $\rho_n\left(f, F_k^{(n)} \backslash C^{(n)}\right) \leq \frac{1}{k^n}$, by virtue of arbitrariness of $f \in C^{(n)}$ and finiteness $C^{(n)}$ we get $h_k(n, C) = \max\left\{ \rho_n\left(f, F_k^{(n)} \backslash C^{(n)}\right) \middle| f \in C^{(n)} \right\} \leq \frac{1}{k^n}$, whence in view of arbitrariness $n \in N$ $H_k(C) = \sup_{n \in N} h_k(n, C) \leq \frac{1}{k}$. So, $H_k(C) = \frac{1}{k}$ for all $C \in \mathcal{R}^*$ (and consequently, for all clones contained in some $C \in \mathcal{R}^*$).

An equality $H_k(C) = \frac{1}{k}$ for clones of the family BO and $Q\mathcal{L}$ is also easy being checked.

For l-aryc $(2 \le l \le k-1)$ central ratio ρ, whose centre $Z(\rho)$ contains the element $a \in E_k$, an equality $H_k(St_k(\rho)) = \frac{l}{k}$ is obtained by owing to constant function c_a^n and that fact that complement of the central relation consists only of vectors with pairwise different coordinates. An equality $H_k(St_k(\rho)) = \frac{q}{k}$ is being established similarly for q-ary $(3 \le q \le k-1)$ ρ ratio which is a strong homomorphic preimage of elementary q-adic ratio. It's easy to show, that set $\left(Z^{(\ge 2)} \cup SH\right)$-of localized subclones $C < \mathcal{F}_k$ is at most countable. The theorem is proved.

The corollary is proved by using Russo's theorem for clones from family $Z^{(\ge 2)}$ (or SH) and absence in minimum clone different from Sel_k proper subclones.

For any $a \ne b$ from E_k we'll consider binary central ratio $\gamma_{a,b} := E_k^2 \backslash \{(a,b),(b,a)\}$ and we'll suppose $\Gamma_{a,b} := St_k\left(\gamma_{a,b}\right)$ ..

Theorem 5. *At any odd $k \ge 3$ and any $a \ne b$ from E_k*

$$H'_k(\Gamma_{a,b}) = \frac{1}{2}$$

is fulfilled.

The proof scheme is similar to the scheme of proof of theorem 3, but slightly differs from it: instead of sets $B_J^a(f)$, $B_J^{E_k \backslash \{a\}}(f)$ we should consider full preimages of all elements from E_k, i.e. set $K_\alpha(f) := \{\tilde{\alpha} \in E_k^n | f(\tilde{\alpha}) = d\}$ $(\alpha \in E_k)$, moreover, here we have to use oddness of $k \ge 3$ as by established of upper estimation $\left(\le \frac{1}{2}\right)$, as by reducing to the contradiction of supposition on it strictness $\left(< \frac{1}{2}\right)$, whereas in the proof of theorem 3 this has been done only in the second case.

Now little bit detailed.

Let's consider arbitrary $f \in F_k^{(n)} \backslash \Gamma_{a,b}^{(n)}$ it's obvious $K_a(f) \ne \varnothing$ and $K_b(f) \ne \varnothing$ however on $\alpha \in E_k \backslash \{a,b\}$ it's possible $K_\alpha(f) = \varnothing$. For this f let's assume

$$K^*(f) := \begin{cases} K_a(f), & \text{if } \quad |K_a(f)| \le |K_b(f)|, \\ K_b(f) & \text{otherwise} \end{cases}$$

$$K^{**}(f) := E_k^n \backslash K^*(f)$$

and some fixed $\alpha \in E_k \backslash \{a,b\}$, let's consider a function

$$g_f(\tilde{x}) := \begin{cases} f(\tilde{x}), & \text{if } \tilde{x} \in K^{**}(f), \\ \alpha, & \text{if } \tilde{x} \in K^*(f). \end{cases}$$

Then at least one of the sets $K_a(g_f)$, $K_b(g_f)$ is empty and that's why $g_f \in \Gamma_{a,b}^{(n)}$. By construction we have $\rho_n(f, g_f) = \frac{|K^*(f)|}{k^n} \le \left[\frac{k^n - |K_d|}{2}\right] \Big/ k^n$. At $K_\alpha(f) \ne \varnothing$ taking into account oddness of $k \ge 3$ (implying evenness of $k^n - 1$) we get $\left[\frac{k^n - |K_d(f)|}{2}\right] \le \frac{k^n - |K_d(f)|}{2} \le \frac{k^n - 1}{2}$, but at $K_d(f) = \varnothing$ again taking into account oddness of k, we get $\left[\frac{k^n - |K_d(f)|}{2}\right] = \left[\frac{k^n}{2}\right] = \frac{k^n - 1}{2}$, i.e., in both cases we have $\left[\frac{k^n - |K_d(f)|}{2}\right] \le \frac{k^n - 1}{2}$. Therefore $\rho_n(f, g_f) \le \frac{k^n - 1}{2k^n} < \frac{1}{2}$, whence $h'_k(n, \Gamma_{a,b}) < \frac{1}{2}$and $H'_k(\Gamma_{a,b}) \le \frac{1}{2}$. Let's show that strict inequality is impossible. Having assumed the contrary, at sufficient large $n_0 \in N$ we have $h'_k(n_0, \Gamma_{a,b}) < \frac{1}{2} - \frac{1}{k^{n_0}}$. For $J = \{j_1, ..., j_s\} \subset \{1, ..., n_0\}$ (case $J = \varnothing$, i.e., $s = 0$ is not excluded) we'll denote cone $\left\{ (\alpha_1, ..., \alpha_{n_0}) \in E_k^{n_0} \middle| (\forall j \in J)\, \alpha_i = a \;\&\; (\forall i \in \{1, ..., n_0\} \setminus J)\, \alpha_i \in E_k \setminus \{a\} \right\}$ by $E_k^{n_0}(a, J)$. The set of $E_k^{n_0} \setminus \{(a, ..., a)\}$ is being divided into the cones $E_k^{n_0}(a, J)$, where $\varnothing \subseteq J \subset \{1, ..., n_0\}$ and in view of oddness of k, each of this cones consists of even number of vectors. Therefore, the function f with condition $f(a, ..., a) = a$, being equal to a on the half of each cones vectors, and being equal to b on the remaining half doesn't belong to $\Gamma_{a,b}^{n_0}$. It is easy to see that for any $g \in \Gamma_{a,b}^{(n_0)}$ is true $\rho_{n_0}(f, g) \geq \frac{k^{n_0 - 1}}{2k^n} > \frac{1}{2} - \frac{1}{k^{n_0}}$, what contradicts to derived from assumption $H_k'(\Gamma_{a,b}) < \frac{1}{2}$ inequality $h_k'(n_0, \Gamma_{a,b}) < \frac{1}{2} - \frac{1}{k^{n_0}}$. So $H_k'(\Gamma_{a,b}) = \frac{1}{2}$ and the theorem is proved.

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Received September 15, 2003; Revised December 26, 2003.
Translated by Mamedova Sh.N.