Vali M. KURBANOV, Rasim A. SAFAROV

ON UNIFORM CONVERGENCE OF ORTHOGONAL EXPANSIONS IN EIGENFUNCTIONS OF STURM-LIOUVILLE OPERATOR

Abstract

In this paper we investigate absolute and uniform convergence on $\bar{G} = [0,1]$ *of orthogonal expansions of functions from the class* $W_p^1(G)(p \geq 1)$, $G = (0,1)$, *satisfying the condition* $f(0) = f(1) = 0$ *in eigenfunctions of arbitrary self-adjoint expansion of Sturm-Liouville operator.*

$$Lu = -u'' + q(x)u$$

with real-valued potential $q(x) \in L_1(G)$. *The rate of uniform convergence of these expansions to function* $f(x)$ *is established.*

Let us consider on internal $G = (0,1)$ arbitrary self-adjoint expansion of the operator

$$Lu = -u'' + q(x)u$$

with real –valued potential $q(x) \in L_1(G)$.

Suppose that the considered expansion has a discrete spectrum. Denote by $\{u_k(x)\}_{k=1}^{\infty}$ orthonormalized and complete in $L_2(G)$ system of eigenfunctions of this expansion and denote by $\{\lambda_k\}_{k=1}^{\infty}$ the corresponding system of eigenvalues. By the definition $u_k(x)$ and $u_k'(x)$ are absolutely continuous functions on $\bar{G}$, $Lu_k \in L_2(G)$, function $u_k(x)$ almost everywhere on G satisfies the equation $Lu_k = \lambda_k u_k$(see [1], [2]). It follows from the results of the papers [3], [4] that sequence $\{\lambda_k\}_{k=1}^{\infty}$ is bounded below. For the definiteness we assume that $\lambda_k \geq 0, \ k \in N$.

Denote $\mu_k = \sqrt{\lambda_k}$; and for arbitrary function $f(x) \in L_1(G)$, let us consider partial sum of its orthogonal expansion in the system

$$\{u_k(x)\}_{k=1}^{\infty} : \sigma_\nu(x,f) = \sum_{\mu_k < \nu} f_k u_k(x),$$

where

$$f_k = (f, u_k) = \int_G f(x) u_k(x) dx, \ \nu > 0.$$

Denote by $W_p^1(G)$, $(p \geq 1)$ the set of absolutely continuous on $\bar{G}$ functions $f(x)$ such that $f^{'}(x) \in L_p(G)$, and denote by $H_p^\alpha(G)$, $(p \geq 1, \ 0 < \alpha \leq 1)$ (Nikolsky class) the set of functions $f(x) \in L_p(G)$ satisfying the condition $\omega_p(f,\delta) \leq C(f)\delta^\alpha$, where

$$\omega_p(f,\delta) = \sup_{0<h\leq\delta} \left\{ \int_0^{1-h} |f(x+h) - f(x)|^p \, dx \right\}^{1/p}.$$

The norm of functions $f(x) \in H_p^\alpha(G)$ is defined by the equality

$$\|f\|_p^\alpha = \|f\|_p + \sup_{\delta>0} \frac{\omega_p(f,\delta)}{\delta^\alpha},$$

where $\|.\|_p -$ is norm in $L_p(G)$.

Suppose $R_\nu(x,f) = f(x) - \sigma_\nu(x,f)$.

The main results of this paper are the following theorems.

Theorem 1. *Let function $f(x)$ belong to the class $W_p^1(G)$, $1 < p \le 2$ and satisfy the condition $f(0) = f(1) = 0$. Then orthogonal expansion of the function in the system $\{u_k(x)\}_{k=1}^\infty$ converges absolutely and uniformly on $\bar{G}$, and the following relations hold*

$$f(x) = \sum_{k=1}^\infty f_k u_k(x); \tag{1}$$

$$\max_{x\in\bar{G}} |R_\nu(x,f)| \le const \nu^{-1/q} \|f\|_{W_p^1(G)}; \tag{2}$$

$$\max_{x\in\bar{G}} |R_\nu(x,f)| = O\left(\nu^{-\frac{1}{q}}\right),\ \nu\to\infty, \tag{3}$$

where $q = p/(p-1)$, symbol "O" depends on $f(x)$.

Theorem 2. *Let $f(x) \in W_1^1(G)$, and the conditions $f(0) = f(1) = 0$ and*

$$\sum_{n=1}^\infty n^{-1}\omega_1(f', n^{-1}) < \infty \tag{4}$$

be satisfied. Then expansion of the function $f(x)$ in the system $\{u_k(x)\}_{k=1}^\infty$ converges absolutely and uniformly on $\bar{G}$, equality (1) and the following estimate hold

$$\max_{x\in\bar{G}} |R_\nu(x,f)| \le const\Omega(\nu), \tag{5}$$

where $\Omega(\nu) = \sum\limits_{k=[\nu]}^\infty \frac{\omega_1(f',\ k^{-1})}{k} + \nu^{-1}(\|q\|_1 + 1)\left\|f'\right\|_1.$

Corollary 1. *If $f(x) \in W_1^1(G), f(0) = f(1) = 0$ and $f'(x) \in H_1^\alpha(G), 0 < \alpha \le 1$, then*

$$\max_{x\in\bar{G}} |R_\nu(x,f)| \le const \nu^{-\alpha} \left\|f'\right\|_1^\alpha. \tag{6}$$

Corollary 2. *If $f(x) \in W_1^1(G)$, $f(0) = f(1) = 0$ and for some $\beta > 0$ the estimate*

$$\omega_1(f',\delta) = O\left(\frac{1}{\ln^{1+\beta}\frac{1}{\delta}}\right),\ \delta\to 0$$

holds, then

$$\max_{x\in\bar{G}} |R_\nu(x,f)| = O(\ln^{-\beta}\nu),\ \nu\to\infty. \tag{7}$$

To prove the formulated results we need the following lemmas.

Lemma 1. *Let $f(x)\in W_1^1(G)$ and $f(0)=f(1)=0$. Then*

$$|f_k|\le\frac{const}{\mu_k}\left\{\left|\int_G f'(y)\sin\mu_k y dy\right|+\left|\int_G f'(y)\cos\mu_k y dy\right|+\right.$$
$$\left.+\int_G|q(\xi)|\left|\int_\xi^1 f(y)\sin\mu_k(y-\xi)dy\right|d\xi\right\},\quad \mu_k\ge 1. \tag{8}$$

Proof. Let us write Titchmarsh formula [5], [6] for the eigenfunction $u_k(y)$:

$$u_k(y)=u_k(0)\cos\mu_k y+u_k'(0)\frac{\sin\mu_k y}{\mu_k}+\frac{1}{\mu_k}\int_0^y q(\xi)u_k(\xi)\times$$
$$\times\sin\mu_k(y-\xi)d\xi.$$

Integrating by parts the integral $f_k=(f,u_k)$ and taking into account the condition $f(0)=f(1)=0$, we obtain:

$$f_k=-\frac{u_k(0)}{\mu_k}\int f'(y)\sin\mu_k y dy+\frac{u_k'(0)}{\mu_k^2}\int_G f'(y)\cos\mu_k y dy+$$
$$+\frac{1}{\mu}\int_0 q(\xi)u_k(\xi)\int_\xi^1 f(y)\sin\mu_k(y-\xi)dyd\xi.$$

There taking into account the estimates

$$\max_{x\in\bar G}|u_k(x)|\le const,\qquad \text{(see [1])}$$
$$|u_k'(0)|\le const\mu_k,\qquad \text{(see [2])} \tag{9}$$

we obtain estimate (8). Lemma 1 is proved.

Lemma 2 ([7], [8]). *For the coefficients of Fouries expansion of arbitrary function $g(x)\in L_1(G)$ with respect to the system of eigenfunctions $\{u_k\}_{k=1}^\infty$ of the operator L, the following estimate is valid*

$$|g_k|=|(g,u_k)|\le const\left(\omega_1(g,\mu_k^{-1})+\frac{\|g\|_1}{\mu_k}\right),\ \mu_k\ge 1, \tag{10}$$

where const is independent of the function $g(x)$.

Note that in this lemma it is not necessary that system $\{u_k(x)\}_{k=1}^\infty$ be orthonormalized and complete. Lemma 2 is also valid in the case when only estimate (9) is fulfilled.

Proof of theorem 1. By virtue of estimate (9) it suffices to show that

$$\sum_{k=1}^\infty|f_k|\le const\,\|f\|_{W_p^1(G)}\,.$$

Since for number μ_k condition

$$\sum_{\tau\le\mu_k\le\tau+1} 1 \le const,\ \forall\tau\ge 0 \tag{11}$$

is fulfilled (see [9]), then we obtain that by virtue of the results of the paper [10] systems $\{\cos\mu_k y\}_{k=1}^{\infty}$ and $\{\sin\mu_k y\}_{k=1}^{\infty}$ are Bessel systems in $L_2(G)$. On the other hand, for arbitrary function $g(y)\in L_1(G)$, the following inequalities hold

$$\sup_k |(g,\sin\mu_k y)| \le \|g\|_1\,,\ \sup_k |(g,\cos\mu_k y)| \le \|g\|_1\,.$$

Then by virtue of Riesz-Thorin theorem [11] these systems satisfy the Hausdorf Young inequality:

$$\begin{gathered} \|\,|(\varphi,\sin\mu_k x)|\,\|_{lq} \le const\,\|\varphi\|_p\,, \\ \|\,|(\varphi,\cos\mu_k x)|\,\|_{lq} \le const\,\|\varphi\|_p\,, \end{gathered} \tag{12}$$

where $\varphi(x)\in L_p(G),\ 1<p\le 2,\ \dfrac{1}{p}+\dfrac{1}{q}=1.$

Taking into account estimates (8), (9), (11) and (12), we estimate the sum $\sum\limits_{k=1}^{\infty}|f_k|$:

$$\sum_{k=1}^{\infty}|f_k| = \sum_{0\le\mu_k<1}|f_k| + \sum_{\mu_k\ge 1}|f_k| \le const\,\|f\|_1 \sum_{0\le\mu_k\le 1} 1 +$$

$$+const\left\{\left(\sum_{\mu_k\ge 1}\mu_k^{-p}\right)^{1/p}\left\|\left|(f',\sin\mu_k y)\right|\right\|_{lq} + \left(\sum_{\mu_k\ge 1}\mu_k^{-p}\right)^{1/p}\times\right.$$

$$\left.\times\left\|\left|(f',\cos\mu_k y)\right|\right\|_{lq} + \int_G |q(\xi)|\left[\left(\sum_{\mu_k\ge 1}^{\infty}\mu_k^{-p}\right)^{1/p}\left\|\left|(\tilde{f}'_\xi(y),\sin\mu_k y)\right|\right\|_{lq}\right]d\xi\right\} \le$$

$$\le const\left\{\|f\|_p\,(1+\|q\|_1) + \left\|f'\right\|_p\right\} \le const\,\|f\|_{W_p^1(G)}\,.$$

Note that the function $\tilde{f}_\xi(y)$ is defined by the formula

$$\tilde{f}_\xi(y) = \begin{cases} f(y+\xi), & at\quad 0\le y\le 1-\xi \\ 0, & at\quad 1-\xi<y\le 1. \end{cases}$$

Thus, Fourier series of function $f(x)$ absolutely and uniformly converges on $\bar{G}$. Since system $\{u_k(x)\}_{k=1}^{\infty}$ is complete orthonormallized, then we have that Fouirer series of function $f(x)$ converges to $f(x)$. Equality (1) is proved.

Now we prove estimates (2) and (3). By virtue of (8), (9), (11) we find:

$$|R_\nu(x,f)| = \left|\sum_{\mu_k\geq\nu} f_k u_k(x)\right| \leq const \sum_{\mu_k\geq\nu} |f_k| \leq$$

$$\leq const \left(\sum_{\mu_k\geq\nu} \mu_k^{-p}\right)^{1/p} \left[\left(\sum_{\mu_k\geq\nu} \left|(f', \sin\mu_k y)\right|^q\right)^{1/q} + \right.$$

$$\left. + \left(\sum_{\mu_k\geq\nu} \left|(f', \cos\mu_k y)\right|^q\right)^{1/q} + \int_G |q(\xi)| \left(\sum_{\mu_k\geq\nu} \left|(\tilde{f}_\xi(y), \sin\mu_k y)\right|^q\right)^{1/q} d\xi\right] \leq$$

$$\leq const\nu^{-\frac{1}{q}}[...], \quad x\in G.$$

Subject to inequalities (12) this implies estimate (2). Besides the first two addends in the brackets are equal to $O(1)$ as $\nu\to\infty$ because estimates (12) hold. Since $\left|(\tilde{f}_\xi(y), \sin\mu_k y)\right| = O\left(\mu_k^{-1}\right)$, then we have that the third term in the brackets is bounded above by the quantity $O(\nu^{-\frac{1}{p}})$. Thus, in brackets we have $O(1), \nu\to\infty$. Consequently, estimate (3) holds.

Theorem 1 is proved.

Proof theorem 2. Since functions $\sin\mu_k x$ and $\cos\mu_k x$, $k=1,2,...$ are eigenfunctions of operator $L_0 = -\dfrac{d^2}{dx^2}$, then we have that subject to (10)

$$\int_G f'(y)\sin\mu_k y dy \leq const\left(\omega_1(f', \mu_k^{-1}) + \frac{\|f'\|_1}{\mu_k}\right),$$

$$\int_G f'(y)\cos\mu_k y dy \leq const\left(\omega_1(f', \mu_k^{-1}) + \frac{\|f'\|_1}{\mu_k}\right), \ \mu_k\geq 1.$$

On the other hand,

$$\left|\int_\xi^1 f(y)\sin\mu_k(y-\xi)dy\right| \leq \frac{1}{\mu_k}\left\{\max|f(x)| + \left\|f'\right\|_1\right\} \leq \frac{2}{\mu_k}\left\|f'\right\|_1,$$

because $f\in W_1^1(G)$, $f(0)=f(1)=0$.

Taking into account these inequalities in (8), we obtain:

$$|f_k| \leq \frac{const}{\mu_k}\left\{\omega_1(f', \mu_k^{-1}) + (1+\|q\|_1)\,\mu_k^{-1}\left\|f'\right\|_1\right\}. \qquad (13)$$

Subject to condition (4) and estimates (9), (11) and (13) we obtain:

$$\sum_{k=1}^{\infty} |f_k u_k(x)| \leq const \left(\sum_{0\leq\mu_k\leq 1} 1 \right) \|f\|_1 + const \sum_{\mu_k \geq 1} \frac{\omega_1(f', \mu_k^{-1})}{\mu_k} +$$
$$+const \left(\sum_{\mu_k\geq 1} \mu_k^{-2} \right) (1+\|q\|_1) \|f'\|_1 \leq const \|f\|_1 +$$
$$+const \sum_{n=1}^{\infty} \left(\sum_{n\leq\mu_k<n+1} \frac{\omega_1(f', \mu_k^{-1})}{\mu_k} \right) + const \|f\|_{W_1^1(G)} \leq$$
$$\leq const \|f\|_{W_1^1(G)} + const \sum_{n=1}^{\infty} \frac{\omega_1(f', n^{-1})}{n} \left(\sum_{n\leq\mu_k\leq n+1} 1 \right) \leq$$
$$\leq const \left\{ \|f\|_{W_1^1(G)} + \sum_{n=1}^{\infty} \frac{\omega_1(f', n^{-1})}{n} \right\} < \infty.$$

Consequently, Fourier series of the function $f(x)$ absolutely and uniformly converges on $\bar{G}$. Since system $\{u_k(x)\}_{k=1}^{\infty}$ is complete and orthonormalized, then we have that this series converges to $f(x)$.

Let us prove estimate (5). Subject to (9), (11) and (13) for any $x \in \bar{G}$, we have:

$$|R_\nu(x,f)| \leq \sum_{\mu_k\geq\nu} |f_k u_k(x)| \leq const \sum_{\mu_k\geq\nu} |f_k| \leq$$

$$\leq const \left\{ \sum_{\mu_k\geq\nu} \frac{\omega_1(f', \mu_k^{-1})}{\mu_k} + (1+\|q\|_1) \|f'\|_1 \sum_{\mu_k\geq\nu} \mu_k^{-2} \right\} \leq$$

$$\leq const \left\{ \sum_{n=[\nu]}^{\infty} \left(\sum_{n\leq\mu_k<n+1} \frac{\omega_1\left(f', \mu_k^{-1}\right)}{\mu_k} \right) + \right.$$

$$\left. + (1+\|q\|_1) \|f'\|_1 \sum_{n=[\nu]}^{\infty} \left(\sum_{n\leq\mu_k<n+1} \mu_k^{-2} \right) \right\} \leq$$

$$\leq const \left\{ \sum_{n=[\nu]}^{\infty} \frac{\omega_1\left(f', n^{-1}\right)}{n} \left(\sum_{n\leq\mu_k<n+1} 1 \right) + \right.$$

$$\left. +(1+\|q\|_1) \|f'\|_1 \sum_{n=[\nu]}^{\infty} \frac{1}{n^2} \left(\sum_{n\leq\mu_k<n+1} 1 \right) \right\} \leq$$

$$\leq const \left\{ \sum_{n=[\nu]}^{\infty} \frac{\omega_1\left(f', n^{-1}\right)}{n} + (1+\|q\|_1) \|f'\|_1 \nu^{-1} \right\} \leq const\Omega(\nu).$$

Theorem 2 is proved.

Note that the obtained results amplify the earlier proved results of the papers [12], [13].

References

[1]. Il'in V.A., Yoó I. *Uniform estimate of eigenfunctions and above estimate of the number of eigenvalues of Sturm-Liouville operator with potential from the class L_p.* Diff. uravnenija, 1979, v.15, No5, pp.1164-1174. (Russian)

[2]. Il'in V.A., Yoó I. *Estimate of difference of partial sums of expansions to two arbitrary nonnegative self-adjeint extensions of two Sturm-Liouville operators for absolutely continuous function.* Diff. uravnenija, 1979, v.15, No7, pp.1175-1193. (Russian) [3]. Volkov V.E. *On boundedness of the number of orthogonal solutions of equation $-u'' + q(x)u = \lambda u$ for large values of $-\lambda$.* Mat. zametki, 1984, v. 36, No5, pp.691-695. (Russian)

[4]. Kerimov N.B. *To the question on necessary conditions of basisness.* Diff. uravnenija, 1990, v.26, No6, pp.943-953. (Russian)

[5]. Titchmarsh E.Ch. *Expansions in eigenfunctions connect with the second order differential equations.* M.: IL, 1960, v.I. (Russian)

[6]. Tikhomirov V.V. *Exact estimates of eigenfunctions of arbitrary non-self-adjoint Shrōdinger operator.* Diff. uravnenija, 1983, v.19, No8, pp.1378-1385. (Russian)

[7]. Kurbanov V.M. *On rate of equiconvergence of partial sums of biorthogonal expansions responding to two differential operators.* Spectr teorija operatorov i prilozh., Baku, 1997, issue XI, pp.99-116. (Russian)

[8]. Kurbanov V.M. *Equiconvergence of biorthogonal expansions in root functions of differential operators.* II Diff. uravnenija, 2000, v.36, No3, pp.319-335. (Russian)

[9]. Kurbanov V.M. *On distribution of eigenvalues of second order differential operator.* Izv. AN Azerb. 1997, v.18, No4-5, pp.106-112. (Russian)

[10]. Il'in V.A. *On unconditioned basicity on of systems of eigen and joined functions of second order differential operators a closed interval.* Soviet Math. Dokl., 1983, v.273, No5, pp.1048-1053. (Russian)

[11]. Zygmund A. *Trigonometric series.* M.: 1965, v.2, 537 p. (Russian)

[12]. Lažetić N.L. *On existence of classical solution of mixed problem for one-dimensional hyperbolic equation of the first order.* Diff.uravnenija, 1998, v.34, No5, pp.682-694. (Russian)

[13]. Lažetić N.L. *On inform convergence of spectral exponsions of an one-dimensional Shrōdinger operator.* Matem.Vestnik (Belgrad), 1999, No51, pp.97-110.

Vali M. Kurbanov, Rasim A. Safarov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 394 720(off.)

Received March 17, 2003; Revised October 27, 2003.

Translated by Azizova R.A.