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ON APPROXIMATE SOLUTION OF ONE CLASS OF BOUNDARY INTEGRAL EQUATIONS

Abstract

In the paper the basis of collocation method is given for singular integral equation of the following form

$$\rho(x)+\int_S \frac{q(x,y)}{|x-y|^2}\rho(y)d\sigma_y = f(x),$$

where S, *is a closed smooth surface in* R^3, $q\in C(S\times S)$, *and* $q(x,x,)=0$ *when* $x\in S$, $f\in C(S)$ *and* ρ *is the unknown function from* $C(S)$.

The solutions of many theoretical and applied problems of mathematics, mechanics and physics lead to the class of boundary integral equations (B and E).

$$\rho(x)+\int_S \frac{q(x,y)}{|x-y|^2}\rho(y)d\sigma_y = f(x), \tag{1}$$

where S is a closed smooth surface in R^3, $q\in C(S\times S)$ and $q(x,x,)=0$ at $x\in S$, $f\in C(S)$, ρ is the unknown function from $C(S)$. It is known that the equation of form (1) can be solved analytically only in very rare cases. Therefore, both for theory and in particular for the development of approximate methods of solution of integral equations with corresponding theoretical basis is very important. The present paper is devoted to a basis of collection method for Boubdary Integral Equations of form (1).

Let $\omega_\rho(\delta)$ be "corrected" module of the continuity of the function ρ, and

$$\omega_q^*(\delta)=\delta\sup_{\tau\geq\delta}\frac{\bar{\omega}_q^*(\tau)}{\tau},\ \delta>0,$$

where

$$\bar{\omega}_q^*(\delta)=\sup_{\substack{|x-y|\leq\delta\\ x,\ y\in S}}|q(x,y)|.$$

Let's introduce the following notation:

$$\omega_q^{1,0}(\delta)=\sup_{\substack{|x_1-x_2|\leq\delta,\\ x_1,\ x_2\in S}}\max_{y\in S}|q(x_1,y)-q(x_2,y)|;$$
$$\omega_q^{0,1}(\delta)=\sup_{\substack{|y_1-y_2|\leq\delta,\\ y_1,\ y_2\in S}}\max_{x\in S}|q(x,y_1)-q(x,y_2)|;$$

$$\|q(x,y)\|_\infty=\max_{x,y\in S}|q(x,y)|.$$

It is evident, that

$$\omega_\rho,\omega_q^*\in\varepsilon_1=$$

$$= \left\{ \varphi \left| \varphi \geq 0, \ \varphi \uparrow, \ \varphi(\delta)/\delta \downarrow, \ \lim_{\delta \to 0} \varphi(\delta) = 0, \ \varphi(\delta_1 + \delta_2) \leq \varphi(\delta_1) + \varphi(\delta_2) \right. \right\};$$

and

$$\omega_q^{1,0}, \omega_q^{0,1} \in \mathcal{E}_2 = \left\{ \varphi \left| \varphi \geq 0, \ \varphi \uparrow, \ \lim_{\delta \to 0} \varphi(\delta) = 0 \right. \right\}.$$

The following theorem holds.

Theorem 1. *Let a)* $\omega_q^{1,0}(\delta) = 0(\ln^{-1}\delta)$ *and b)* $\int\limits_0^{diam\ S} \frac{\omega_q^*(\tau)}{\tau} d\tau < +\infty$.

Then, the integral operator

$$(K\rho)(x) = \int\limits_S \frac{q(x,y)}{|x-y|^2} \rho(y) d\sigma_y$$

in compact on $C(S)$.

Proof. In [1] it is shown, that under the conditions a) and b) the integral

$$W(x) = \int\limits_S \frac{q(x,y)}{|x-y|^2} \rho(y) d\sigma_y \qquad (2)$$

converges as singular and $W \in C(S)$, i.e. $K : C(S) \to C(S)$. For proof of compactness of the operator K on $C(S)$ we introduce the continuous functions $Q_n : S \times S \to C, \ n \in N$ of the form

$$Q_n(x,y) = \begin{cases} 0, \ \text{if} \ |x-y| \leq \frac{1}{2n}, \\ \frac{(2n|x-y|-1) \cdot q(x,y)}{|x-y|^2}, \ \text{if} \ \frac{1}{2n} \leq |x-y| \leq \frac{1}{n}, \\ \frac{q(x,y)}{|x-y|^2}, \ \text{if} \ |x-y| \geq \frac{1}{n}. \end{cases}$$

Let $S_\delta(x) = \{y \in S | \ |x-y| \leq \delta\}$ and

$$(G_n\rho)(x) = \int\limits_S Q_n(x,y)\rho(y)d\sigma,$$

then

$$|(K\rho)(x) - (G_n\rho)(x)| \leq \int\limits_{S_{1/n}(x)} \left| \frac{q(x,y)}{|x-y|^2} \rho(y) \right| d\sigma_y \leq$$

$$\leq \|\rho\|_\infty \int\limits_0^{2\pi} \int\limits_0^{1/n} \frac{\omega_q^*(\tau)}{\tau^2} \tau d\tau \ \ d\alpha = 2\pi \|\rho\|_\infty \int\limits_0^{1/n} \frac{\omega_q^*(\tau)}{\tau} d\tau.$$

From the last estimation we obtain, that $\|K - G_n\|_\infty \to 0$ as $n \to \infty$.

Besides taking into account the compactness of the operators G_n on $C(S)$ we obtain, that the operator K compact on $C(S)$, what proves the theorem.

Now we'll give the basis to the collection method for BIE of form (1). Let $\{h\} \subset R_+$ be a set of values of discretization parameter tending to zero. Divide S into a "regular" elementary domains:

$$S = \bigcup_{l=1}^{N(h)} S_l^h$$

(see [2]), and let $x_l \in S_l^h,\ l=\overline{1,N(h)}$ be support points.

We'll take the expression

$$(K^{N(h)}\rho)(x_l)=\sum_{\substack{j=1,\\ j\neq l}}^{N(h)}\frac{q(x_l,x_j)}{|x_l-x_j|^2}\rho(x_j)mesS_j^h \tag{3}$$

as cubature formula (c.f.) for integral (2) at the points $x_l,\ l=\overline{1,N(h)}$. For the estimation of error of c.f. (3) the following is true.

Theorem 2. [3]. *Let the conditions a), b) and c) $\omega_q^{0,1}(\delta)=0(\ln^{-1}\delta)$ be fulfilled. Then*

$$\max_{l=\overline{1,N(h)}}\left|(K\rho)(x_l)-(K^{N(h)}\rho)(x_l)\right|\leq$$

$$\leq M\left(\omega_\rho(R(h))+\|\rho\|_\infty\left(\omega_q^{0,1}(R(h))\left|\ln R(h)\right|+\right.\right.$$

$$\left.\left.+\int_0^{R(h)}\frac{\omega_q^*(\tau)}{\tau}d\tau+R(h)\int_{r(h)}^{diam\ S}\frac{\omega_q^*(\tau)}{\tau^2}d\tau\right)\right),$$

holds, where M is a positive constant depending only on S and

$$R(h)=\max_{l=\overline{1,N(h)}}\sup_{x\in\partial S_l^h}|x-x_l|\,,\ r(h)=\min_{l=\overline{1,N(h)}}\inf_{x\in\partial S_l^h}|x-x_l|\,.$$

Denote by $C^{N(h)}$ the space of $N(h)$-dimensional vectors

$$w^{N(h)}=(w_1,w_2,...,w_{N(h)}),\ w_t\in\mathbb{C},l=\overline{1,N(h)}$$

with the norm $\left\|w^{N(h)}\right\|=\max\limits_{l=1,N(h)}|w_l|$. Assume $w^{N(h)}\in\mathbb{C}^{N(h)}$ for

$$K_l^{N(h)}w^{N(h)}=\sum_{\substack{j=1,\\ j\neq l}}^{N(h)}\frac{q(x_l,x_j)}{|x_l-x_j|^2}mesS_j^h\cdot w_j,\ l=\overline{1,N(h)}\ ;$$

$$K^{N(h)}w^{N(h)}=(K_1^{N(h)}w^{N(h)},..,K_{N(h)}^{N(h)}w^{N(h)}).$$

According to the collection method with using c.f. (3), BIE is replaced by a system of algebraic equations relative to w_l approximate values $\rho(x_l),\ l=\overline{1,N(h)}$ which we'll write in the following form:

$$w^{N(h)}+K^{N(h)}w^{N(h)}=f^{N(h)}, \tag{4}$$

where

$$f^{N(h)}=p^{N(h)}f=(f_1,...,f_{N(h)});\ f_l=f(x_l),\ l=\overline{1,N(h)};\ p^{N(h)}\in\mathcal{L}(C(S),\ \mathbb{C}^{N(h)})$$

is operator of a simple drift.

Theorem 3. *Under the conditions a), b), c) and* $Ker(I+K)=\{0\}$ *equations (1) and (4) have unique solutions* $\rho_* \in C(S)$ *and* $w_*^{N(h)} \in \mathbb{C}^{N(h)}$ $(N(h) \geq n_0)$ *moreover* $\left\| w_*^{N(h)} - p^{N(h)}\rho_* \right\| \to 0$ *as* $h \to 0$ *with the estimation*

$$C_1\delta_{N(h)} \leq \left\| w_*^{N(h)} - p^{N(h)}\rho_* \right\| \leq C_2\delta_{N(h)},$$

where

$$C_1 = \left(\sup_{N(h)\geq n_0} \left\| I^{N(h)} + K^{N(h)} \right\| \right)^{-1} > 0,$$

$$C_2 = \sup_{N(h)\geq n_0} \left\| \left(I^{N(h)} + K^{N(h)} \right)^{-1} \right\| < \infty,$$

$$\delta_{N(h)} = \max_{l=\overline{1,N(h)}} \left| K_l^{N(h)}(p^{N(h)}\rho_*) - (K\rho_*)(x_l) \right|,$$

and $I^{N(h)}$ *is a unit operator in* $\mathbb{C}^{N(h)}$.

Proof. Apply G.M. the Vaynikko convergence theorem on for linear operator equations (see [4]).

Let $\mathcal{P}=\{p^{N(h)}\}$ be system of operators of simple drift. If is obvious, that the system $\mathcal{P}$ is a index for $C(S)$ and $\mathbb{C}^{N(h)}$, $N(h)=1,2,....$ Besides $f^{N(h)} \xrightarrow{\mathcal{P}} f$ when $h \to 0$ and the operators $I^{N(h)} + K^{N(h)}$ are fredholmian with null index.

Now we show, that $K^{N(h)} \xrightarrow{\mathcal{PP}} K$ is compact. Since $K^{N(h)} \xrightarrow{\mathcal{PP}} K$, then by the known proposal (see [4], p.8) it is sufficient to prove, that there exists relatively compact sequence $\{K_{N(h)}w^{N(h)}\} \subset C(S)$ where $w^{N(h)} \in \mathbb{C}^{N(h)}$ and $\left\|w^{N(h)}\right\| \leq C = const$ such that $\left\|K^{(N(h)}w^{N(h)} - p^{N(h)}(K_{N(h)}w^{N(h)})\right\| \to 0$ when $h \to 0$. As $\{K_{N(h)}w^{N(h)}\}$ we choose the sequence. If it is not difficult to understand, that

$$(K_{N(h)}w^{N(h)})(x) = \sum_{j=1}^{N(h)} \left[\int_{S_j^h} \frac{q(x,y)}{|x-y|^2} d\sigma_y \right] \cdot w_j, \ x \in S.$$

Since $\{K_{N(h)}w^{N(h)}\}\subset C(S)$, we obtain that

$$\left|K_l^{N(h)}w^{N(h)}-(K_{N(h)}w^{N(h)})(x_l)\right|=\left|\sum_{\substack{j=1,\\ j\neq l}}^{N(h)}w_j\left[\int_{S_j^h}\left(\frac{q(x_l,x_j)-q(x_l,y)}{|x_l-y|^2}\right)d\sigma_y+\right.\right.$$

$$\left.\left.+\int_{S_j^h}q(x_l,x_j)\left(\frac{1}{|x_l-x_j|^2}-\frac{1}{|x_l-y|^2}\right)d\sigma_y\right]+w_l\int_{S_j^h}\frac{q(x_l,y)}{|x_l-y|^2}d\sigma_y\right|\leq$$

$$\leq const\left[\int_{S/S_j^h}\frac{\omega^{0,1}(R(h))}{|x_l-y|^2}d\sigma_y+R(h)\int_{S/S_j^h}\frac{\omega_q^*(|x_l-y|)}{|x_l-y|^3}d\sigma_y+\right.$$

$$\left.+\int_{S_j^h}\frac{\omega_q^*(|x_l-y|)}{|x_l-y|^2}d\sigma_y\right]\leq const\times$$

$$\times\left[\int_0^{R(h)}\frac{\omega_q^*(\tau)}{\tau}d\tau+\omega_q^{0,1}(R(h))\left|\ln R(h)\right|+R(h)\int_{r(h)}^{diam\ S}\frac{\omega_q^*(\tau)}{\tau^2}d\tau\right],$$

when

$$\left\|K^{(N(h)}w^{N(h)}-p^{N(h)}(K_{N(h)}w^{N(h)})\right\|\to 0.$$

And the relative compactness $\{K_{N(h)}w^{N(h)}\}$ follows from Arcel's theorem (see [5]).

The theorem is proved.

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Received June 17, 2003; Revised November 10, 2003.
Translated by Javadova I.V.