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ON M.V.KELDYSH MULTIPLE COMPLETENESS OF A SYSTEM OF ROOT VECTORS OF THE HIGHER ORDER OPERATOR BUNDLE

Abstract

In this paper the theorem on multiple completeness of a system of derived chain in M.V.Keldysh sense for a class of the higher order operator bundles is proved.

As is known, the investigation of completeness of a system of root vectors of operator bundles is one of more important problems in the spectral theory of operator bundles. In 1951 at first the concept of n -fold completeness of a system of eigen and adjoined vectors has been given and the fundamental theorem on multiple completeness of these vectors for some classes of operator bundles has been proved in Keldysh's paper [1].

Later on, M.V.Keldysh's theory has been generalized and developed in the papers of J.E.Allahverdiyev [2], M.G.Gasymov [3], G.V.Radzievsky [4], R.M.Jabarzade [5] and others.

In this paper n-fold completeness of eigen and adjoined vectors of operator bundle of the form

$$A(\lambda) = \sum_{j=0}^{n} \lambda^j A_j$$

is investigated where λ is a complex parameter, A_j $(j = 0, 1, ..., n)$ are linear operators acting in the Hilbert space H.

We lead the definition which will be used later on.

Assume $L(\lambda) = E - A(\lambda)$.

Definition 1. *The number* λ_0 *is called an eigen element of the operator bundle* $L(\lambda)$, *if there exists nonzero vector* $\varphi_0 \in H$ *such that the equality*

$$L(\lambda_0)\varphi_0 = 0$$

is satisfied.

At that φ_0 is called an eigen vector of the bundle $L(\lambda)$ respondent to the eigen element λ_0. If $\varphi_1, \varphi_2, ..., \varphi_k$ satisfy the equations

$$\sum_{p=0}^{j} \frac{1}{p!} L^{(p)}(\lambda_0)\varphi_{j-p} = 0 \qquad (j = 1, 2, ..., k),$$

then we call them the chain of the vectors adjoined to the eigen vector φ_0 of the bundle $L(\lambda)$.

Let φ_0 be an eigen vector of the bundle $L(\lambda)$ respondent to the eigen value λ_0, and $\varphi_1, ..., \varphi_k$ be corresponding adjoined elements determining the adjoined chain of elements. We define the elements $\tilde{\varphi}_s \in H^n$, where H^n is the direct sum of n-th spaces H, by the following form

$$\tilde{\varphi}_s = \left(\varphi_s^{(0)}, \varphi_s^{(1)}, ..., \varphi_s^{(n-1)}\right), \text{ at that } \varphi_0^{(0)} = \varphi_0,\ \varphi_s^{(0)} = \varphi_s \quad (s = 1, ..., k),$$

$$\varphi_0^{(\nu)} = \lambda_0^{\nu}\varphi_0^{(0)} = \lambda_0^{\nu}\varphi_0, \quad \varphi_s^{(\nu)} = \lambda_s\varphi_s^{(\nu-1)} + \varphi_{s-1}^{(\nu-1)} \quad (\nu = 1, ..., k).$$

Definition 2. T*he system of eigen and adjoined vectors is called n-fold complete by M.V.Keldysh in the space H, if the system $\tilde{\varphi}_s$ constructed for all eigen values is complete in H^n.*

(We recall that the class of completely continuous operators is denoted by G_∞. Let $A \in G_\infty$. Then the operator $B = (A^*A)^{1/2} \in G_\infty$.

The eigen values of the operator B is called s-numbers of the operator A.

The completely continuous operator B has the finite order if its s-numbers are such that $\sum\limits_{i=0}^{\infty} |s_i|^{\rho} < \infty$ for any $\rho > 0$..

The lower bound of the numbers $\rho : \rho_0 = \inf \rho$ is the order of the operator B.

Denote by G_ρ a class of completely continuous operators with the finite order ρ. The operator is called complete on Keldysh if it is annulled only to zero).

We now proof the theorem on n-fold completeness of eigen and adjoined vectors of a class of n -th order operator bundle.

Theorem. *Let the following conditions be satisfied in the Hilbert space H*

a) C is a complete, completely continuous, normal operator of the finite order ρ, the spectrum of this operator is on the finite number rays;

b) the operators $A_0,\ A_1, ...,\ A_{n-1}$ are completely continuous;

c) the operators $B_0,\ B_1, ...,\ B_{n-1}$ are bounded, moreover the inequalities

$$\left(\sum_{i=0}^{n-1} \|B_i\|^2\right)^{1/2} < \sin\frac{\pi}{\rho},$$

(where ρ is order of the operator C) are satisfied.

Then the system of eigen and adjoined elements of operator bundle

$$L(\lambda) = (A_0 + B_0) + \lambda(A_1 + B_1)C + ... + \lambda^{n-1}(A_{n-1} + B_{n-1})C^{n-1} + \lambda^n C^n \quad (1)$$

forms n-fold complete system in the space H.

Proof of the theorem.

Let H^n be direct sum of n Hilbert spaces, i.e., be a space whose elements are ordered systems from n elements of the space H. Consider the following operator equation in the Hilbert space H^n

$$\left(\tilde{D}+\lambda\tilde{C}\right)\tilde{x}=\tilde{x}, \tag{2}$$

where $\tilde{D}=\tilde{A}+\tilde{B}$, the operators $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are given with the help of the matrices in direct sums of the spaces H^n

$$\tilde{A}=\begin{pmatrix} A_0 & A_1 \ ... \ A_{n-1} \\ 0 & 0 \ ... \ 0 \\ & .... \\ & .... \\ 0 & 0 \ ... \ 0 \end{pmatrix}$$

$$\tilde{B}=\begin{pmatrix} B_0 & B_1 \ ... \ B_{n-1} \\ 0 & 0 \ ... \ 0 \\ & .... \\ & .... \\ 0 & 0 \ ... \ 0 \end{pmatrix}$$

$$\tilde{C}=\begin{pmatrix} 0 & 0 & ... & 0 & C \\ C & 0 & ... & 0 & 0 \\ 0 & C & ... & 0 & 0 \\ & & .... & & \\ & & .... & & \\ 0 & 0 & ... & C & 0 \end{pmatrix}.$$

The operator $\tilde{A}$ is completely continuous, because the operators A_i $\left(i=\overline{0,n-1}\right)$ are completely continuous; the operator $\tilde{B}$ is bounded by virtue of the boundedness of the operators B_i $\left(i=\overline{0,n-1}\right)$ by the condition of the theorem. The operator $\tilde{C}$ is a complete, completely continuous, normal operator of the finite order ρ, because the operator C is like by condition a) of the theorem.

Thus, $\tilde{D}$ is a continuous operator with the bounded part $\tilde{B}$, the norm of this operator is no more than $\sin\frac{\pi}{\rho}$ by the condition b), where ρ is order of the operator C.

Consequently, all the conditions of theorem 1 of J.E.Allahverdiyev from the paper [2] are satisfied, and the multiple completeness of a system of the root vectors of operator bundle (2) holds.

We now show that association between a system of eigen and adjoined vectors of bundle (1) and a system of eigen and adjoined vectors of equation (2).

Let $\tilde{x}=(x_0,x_1,...,x_{n-1})$ be an eigen element corresponding to the eigen value λ

of the operator bundle $L(\lambda)=\tilde{A}+\tilde{B}+\lambda\tilde{C}$. Then we can write

$$\begin{cases} A_0x_0+A_1x_1+...+A_{n-1}x_{n-1}+B_0x_0+B_1x_1+...+B_{n-1}x_{n-1}+ \\ \qquad\qquad\qquad\qquad\qquad\qquad +\lambda Cx_{n-1}=x_0 \\ x_1=\lambda Cx_0 \\ x_2=\lambda Cx_1 \\ .... \\ .... \\ x_{n-1}=\lambda Cx_{n-2} \end{cases}.$$

After successive substitutions we have

$$x_{n-1}=\lambda Cx_{n-2}=\lambda^2C^2x_{n-3}=...=\lambda^{n-1}C^{n-1}x_0.$$

Consequently,

$$A_0x_0+\lambda A_1Cx_0+\lambda^2A_2C^2x_0+...+\lambda^{n-1}A_{n-1}C^{n-1}x_0+$$

$$+B_0x_0+\lambda B_1Cx_0+...+\lambda^{n-1}B_{n-1}x_0+\lambda^nC^nx_0=x_0$$

or

$$\begin{cases} \left[(A_0+B_0)+\lambda(A_1+B_1)C+...+\lambda^{n-1}(A_{n-1}+B_{n-1})C^{n-1}+\lambda^nC^n\right]x_0=x_0 \\ x_1=\lambda Cx_0 \\ x_2=\lambda Cx_1 \\ .... \\ .... \\ x_{n-1}=\lambda Cx_{n-2} \end{cases} \qquad (3)$$

Now let $\tilde{y}=(y_0,y_1,...,y_{n-1})$ be the first element joint to the eigen element $\tilde{x}$ respondent to the eigen value λ of operator equation (2). Then

$$\tilde{y}=\left(\tilde{A}+\tilde{B}+\lambda\tilde{C}\right)\tilde{y}+\tilde{C}\tilde{x}.$$

Consequently, allowing for (3) we can write

$$\begin{cases} y_0=A_0y_0+A_1y_1+...+A_{n-1}y_{n-1}+B_0y_0+B_1y_1+...+B_{n-1}y_{n-1}+ \\ \qquad\qquad\qquad\qquad\qquad\qquad +\lambda Cy_{n-1}+Cx_{n-1} \\ y_1=\lambda Cy_0+Cx_0 \\ y_2=\lambda Cy_1+Cx_1=\lambda C(\lambda Cy_0+Cx_0)+\lambda C^2x_0=\lambda^2C^2y_0+2\lambda C^2x_0 \\ y_3=\lambda Cy_2+Cx_2=\lambda^3C^3y_0+3\lambda^2C^3x_0 \\ .... \\ .... \\ y_{n-1}=\lambda^{n-1}C^{n-1}y_0+(n-1)\lambda^{n-2}C^{n-1}x_0 \end{cases} \qquad (4)$$

After successive substitution in the first equation all consequents of (4) and grouping according to x_0 and y_0, we obtain

$$y_0 = L(\lambda) y_0 + \frac{\partial}{\partial \lambda} L(\lambda) x_0.$$

Consequently, y_0 is a vector joint to x_0. By the analogously reasoning we can show that if $\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_k$ is a Jordan chain corresponding to the eigen vector $\tilde{x}_0$, then we have that the coordinates of these elements form a Jordan chain of bundle (1). Analogously, we can show the reverse one.

Since, single completeness of the system of eigen and adjoined elements of equation (2) in the Hilbert space H^n means n-fold completeness of system of eigen and adjoined elements of operator bundle (1) in the space H, the theorem is proved.

Remark to the theorem.

The proved theorem is also valid, when the characteristic values of operator bundle are inside of arbitrary small angles except for their finite number, whose bisectors are rays coming from the origin of coordinate.

Note that the obtained theorem on n - fold completeness of Keldysh derived chain of operator bundle (1) whose coefficients satisfy the conditions of the above proved theorem, when $B_0, B_1, ..., B_{n-1}$ are operators bounded in H, with sufficiently small norms, depending on the order of the operator C. As it is obvious at $B_1 = B_2 = ... = B_{n-1} = 0$ we arrive at theorem 1 of J.E.Allahverdiyev from the paper [2].

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