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TO THE M. RIESZ THEOREM ON ABSOLUTE CONVERGENCE OF THE TRIGONOMETRIC FOURIER SERIES

Abstract

In this paper we investigate a problem on spread of the known M. Riesz criterion for absolute convergence of the trigonometric Fourier series of continuous functions for the value $p \neq 2$. In particular, the upper estimates of the quantity characterizing the velocity of absolute convergence of Fourier series of the convolution of two functions by product of the best approximations of these functions are obtained. The exactness of corresponding estimates in the scale of power majorants is proved in the sense of the order.

Let $L_p(T)$, $1 \le p < \infty$, be the space of all measurable 2π periodic functions $f : R \to \mathbf{C}$ with the finite norm

$$\|f\|_p = \left((1/2\pi)\int_T |f(x)|^p\,dx\right)^{1/p} < \infty,$$

$C(T) \equiv L_\infty(T)$ be the space of all continuous 2π periodic functions, $\|f\|_\infty = \max\{|f(x)|;\ x \in T\}$, where $T = [-\pi, \pi]$.

For a function $f \in L_1(T)$ with the Fourier-Lebesgue series

$$f(x) \sim \sum_{n\in Z} c_n(f)\ e^{inx}, \quad x \in T, \tag{1}$$

denote $\rho_n^{(\gamma)}(f) = \left(\sum_{|\nu|=n+1}^{\infty} |c_\nu(f)|^\gamma\right)^{1/\gamma}$, $\gamma \in (0,\infty)$, $n \in Z_+$.

It is obvious that if $\rho_0^{(\gamma)}(f) < \infty$, then $\rho_n^{(\gamma)}(f) \downarrow 0$ $(n \uparrow \infty)$; besides, it is clear that the condition $\rho_0^{(1)}(f) < \infty$ provides for absolute and uniform convergence of series (1) everywhere on T, moreover $\|f(\cdot) - S_n(f;\cdot)\|_\infty \le \rho_n^{(1)}(f)$, where $S_n(f;x)$ are partial sums of series (1) of order $n \in Z_+ : S_n(f;x) = \sum_{|\nu|=0}^{n} c_\nu(f)\,e^{i\nu x}$. It is also obvious that from the absolute convergence of series (1) everywhere on T it follows that $\rho_0^{(1)}(f) < \infty$.

The convolution $h = f * g$ of the functions $f \in L_1(T)$ and $g \in L_1(T)$ is defined by the following formula

$$h(x) = (f * g)(x) = (2\pi)^{-1} \int_T f(x-y)\,g(y)\,dy. \tag{2}$$

It is clear that (see for example, [1], v.1, §2.1, p.64-65, [2], v.1, §3.1, p.65-66) the function h is determined everywhere, is 2π periodic, measurable and $\|h\|_1 \le \|f\|_1 \times \|g\|_1$, whence in particular it follows that $h = f*g \in L_1(T)$. The last assertion is a special case of the following result known as W.Young inequality (see for example, [1], v.1, theorem (1.15), pp.67-68; [2], v.2, theorem 13.6.1, pp.176-177; [2], v.1, theorem 3.1.4, p.70, theorem 3.1.6, p.72): let $1 \le p \le \infty$, $1 \le q \le \infty$, $1/r = 1/p+1/q-1 \ge 0$;

then for any functions $f \in L_p(T)$ and $g \in L_q(T)$, their convolutions $h = f * g \in L_r(T)$ and $\|h\|_r \le \|f\|_p \cdot \|g\|_q$. When $1/p + 1/q = 1$, i.e., $q = p'$ is an index adjoint to p ($p' = 1$ for $p = \infty$ and $p' = \infty$ for $p = 1$), the convolution $h = f * g$ is determined everywhere, continuous and $\|h\|_\infty \le \|f\|_p \cdot \|g\|_{p'}$.

We also note that the Fourier coefficients $c_n(h)$ of the convolution $h = f * g$ of two functions $f \in L_1(T)$ and $g \in L_1(T)$ are calculated by the formula (see [1], v.1, theorem (1.5), p.64; [2], v.1, p.66, formula (3.1.5))

$$c_n(h) \equiv c_n(f * g) = c_n(f) \cdot c_n(g), \ n \in Z, \tag{3}$$

so that

$$h(x) \ \sim \ \sum_{n \in Z} c_n(f) \ c_n(g) \ e^{inx}, \quad x \in T; \tag{4}$$

in particular, if $f \in L_2(T)$, $g \in L_2(T)$, then one can replace in (4) " $\sim$ " by " $=$ ".

Denote by $A^{(\gamma)}(T)$ the class of all the functions $f \in L_1(T)$ for which $\rho_0^{(\gamma)}(f) < \infty \quad \left(A^{(1)}(T) \equiv A(T)\right)$.

The following statement belongs to M.Riesz and is noted in the paper [3] (see [4], §9.7, p.634-635; [1], v.1, ch.6, theorem 6 in p.399; [5], § 2.2, p.17; [2], v.1, §10.6.2, remark (4) in p.208).

Theorem A (M.Riesz criteria of absolute convergence of trigonometric Fourier series of continuous functions).

1) The convolution $h = f * g$ of any two functions $f \in L_2(T)$ and $g \in L_2(T)$ belongs to the class $A(T)$;

2) Each function $h \in A(T)$ is representable in the form of the convolution $h = f * g$ of two functions $f \in L_2(T)$ and $g \in L_2(T)$.

Remark 1. 1) In the case $2 < p \le \infty$ the convolution $h = f * g$ of any two functions $f \in L_p(T)$ and $g \in L_p(T)$ belongs to the class $A(T)$. This obvious assertion is a consequence of monotonicity of L_p - norm: $\|f\|_2 \le \|f\|_p$, $\|g\|_2 \le \|g\|_p$ by virtue of which $L_p(T) \subset L_2(T)$, and consequently, $f \in L_2(T)$ and $g \in L_2(T)$.

2) In the case $1 \le p < 2$ the corresponding assertion does not hold, more exactly: for each $p \in [1, 2)$ there exist the functions $f_0(\cdot\, ;\, p) \in L_p(T)$, $g_0(\cdot\, ;\, p) \in L_p(T)$ such that their convolution $h_0 = f_0 * g_0 \notin A(T)$ (see below Example 1 (case $p = 1$) and Example 2 (case $1 < p < 2$)).

As for point 2) of Remark 1 we note that the following theorem is proved in the paper [6], theorem 4A in p.53.

Theorem B. *If the functions $f \in L_p(T)$ and $g \in L_p(T)$ for some $p \in (1, 2]$, then their convolution $h = f * g \in A^{(p'/2)}(T)$, where $p' = p/(p-1)$.*

The proof of this theorem based on sequential application of the Young's inequality (see for example, [1], v.1, §1.9, inequality (9.2) in p.34): $ab \le (1/p)\, a^p + (1/p')\, b^{p'}$ $(a, b \ge 0)$ in the case $p = p' = 2$ and based on the first part of the Hausdorff-Young theorem (see for example, [1], v.2, §12.2, theorem (2.3) in p.153; [2], v.2, §13.5, theorem 13.5.1 in p.172; [4], §2.4, p.211) leads to the estimation

$$\sum_{n \in Z} |\ c_n(h)|^{p'/2} \le (1/2) \left\{ \|f\|_p^{p'} + \|g\|_p^{p'} \right\} < \infty. \tag{5}$$

In the paper [6] (p.53, theorem 5) it was proved that the statement of Theorem B is exact, namely, for each $p \in (1, 2]$ there exist the functions $f_0(\cdot\, ;\, p) \in L_p(T)$,

$g_0(\cdot\ ;\ p) \in L_p(T)$ such that for their convolution $h_0 = f_0 * g_0$ the series in the left hand side of (5) diverges for any number $\beta < p'/2$, i.e., one cannot take a lower index $p'/2 \geq 1$ in (5) (see below Example 3). Consequently, since $p'/2 > 1$ for $1 < p < 2$ then a fortiori $h_0 = f_0 * g_0 \notin A(T)$ in the case $p \in (1,2)$ (see Example 2).

Remark 2. The statement of theorem B on convergence of series in the left hand side of (5) under the conditions $f,\ g \in L_p(T)$ and $1 < p \leq 2$ was later on cited (not referring to [6]) in the monograph [2], v.2, p.224, expression 13.20.

We obtain the following assertion which to a certain degree revises and complements the result cited in theorem B.

Theorem 1. *Let* $1 < p \leq 2,\quad f \in L_p(T),\quad g \in L_p(T),\ h = f * g,\ \gamma = p'/2 \in [1,\infty),\ \gamma' = p/(2-p) \in (1,\infty]\quad (1/\gamma + 1/\gamma' = 1)$*; then*

1) $h \in L_{\gamma'}(T)$ *at* $p < 2$ *and* $h \in C(T)$ *at* $p = 2$, *moreover* $\|h\|_{\gamma'} \leq \|f\|_p \cdot \|g\|_p$;

2) $\left(\sum_{n\in Z} |c_n(h)|^\gamma\right)^{1/\gamma} \leq \|f\|_p \cdot \|g\|_p$;

3) $\rho_n^{(\gamma)}(h) = \left(\sum_{|\nu|=n+1}^{\infty} |c_\nu(h)|^\gamma\right)^{1/\gamma} \leq (M(p))^2 \cdot E_n(f)_p \cdot E_n(g)_p,\ n \in Z_+$, *where* $M(p)$ *is the constant in the known M. Riesz inequality (see for example, [4], §8.20, p.594; [2], v.2, §12.10, p.120; [7], §5.11, p.339)*

$$\|\varphi(\cdot) - S_n(\varphi;\cdot)\|_p \leq M(p) \cdot\ E_n(\varphi)_p,\ n \in Z_+, \tag{6}$$

where $1 < p < \infty,\ \varphi \in L_p(T),\ E_n(\varphi)_p$ *is the best approximation of the function* φ *by trigonometric polynomials of order* $\leq n$ *in* $L_p(T)$.

Proof. 1) The statement $h \in L_{\gamma'}(T)$ at $p < 2$ and $h \in C(T)$ at $p = 2$ is an obvious consequence of formulated above W. Young's inequality $\|h\|_r \leq \|f\|_p \cdot \|g\|_q,\ 1/r = 1/p + 1/q - 1 \geq 0$, in which we have to put $p = q$: $r = p/(2-p) = \gamma'$ for $p < 2$ and $r = \infty$ for $p = 2$.

2) By equality (3), Cauchy-Schwartz inequality and mentioned above Hausdorff-Young inequality, we have

$$\sum_{n\in Z} |c_n(h)|^\gamma = \sum_{n\in Z} |c_n(h)|^{p'/2} = \sum_{n\in Z} |c_n(f)|^{p'/2} |c_n(g)|^{p'/2} \leq$$

$$\leq \left(\sum_{n\in Z} |c_n(f)|^{p'}\right)^{1/2} \left(\sum_{n\in Z} |c_n(g)|^{p'}\right)^{1/2} \leq \|f\|_p^{p'/2} \cdot \|g\|_p^{p'/2},$$

whence

$$\left(\sum_{n\in Z} |c_n(h)|^\gamma\right)^{1/\gamma} \leq \|f\|_p \cdot \|g\|_p .$$

3) Fix arbitrary $n \in N$ and denote $(x \in T)$

$$f_n(x) = f(x) - S_n(f;x) \sim \sum_{|\nu|=n+1}^{\infty} c_\nu(f)\, e^{i\nu x},$$

$$g_n(x) = g(x) - S_n(g;x) \sim \sum_{|\nu|=n+1}^{\infty} c_\nu(g)\, e^{i\nu x};$$

then, by (3) and (4), we have

$$h_n(x) = f_n(x) * g_n(x) \sim \sum_{|\nu|=n+1}^{\infty} c_\nu(f)\, c_\nu(g)\, e^{i\nu x} = h(x) - S_n(h;x),$$

and consequently, by inequality 2) and Riesz inequality (6), we have

$$\rho_0^{(\gamma)}(h_n) \equiv \rho_n^{(\gamma)}(h) = \left(\sum_{|\nu|=n+1}^{\infty} |c_\nu(f)\, c_\nu(g)|^\gamma\right)^{1/\gamma} \le$$

$$\le \|f_n(\cdot)\|_p \cdot \|g_n(\cdot)\|_p = \|f(\cdot) - S_n(f;\cdot)\|_p \cdot \|g(\cdot) - S_n(g;\cdot)\|_p \le$$

$$\le M(p)\, E_n(f)_p \cdot M(p)\, E_n(g)_p = (M(p))^2 \cdot E_n(f)_p \cdot E_n(g)_p .$$

Theorem 1 is proved.

Remark 3. Taking the obvious inequality $ab \le (1/2)\left(a^2 + b^2\right) \quad (a, b \ge 0)$, one shows that the upper estimation of the quantity $\sum_{n\in Z} |c_n(h)|^{p'/2}$ obtained in the proof of point 2) of Theorem 1 is more exact than the estimation (5) (see Theorem B): $\sum_{n\in Z} |c_n(h)|^{p'/2} \le \|f\|_p^{p'/2} \cdot \|g\|_p^{p'/2} \le (1/2)\left(\|f\|_p^{p'} + \|g\|_p^{p'}\right)$.

Remark 4. 1) Assuming in 2) of Theorem 1 $p = 2$ ($\Longrightarrow p' = 2 \Longrightarrow \gamma = p'/2 = 1$), one obtains the estimation $\sum_{n\in Z} |c_n(h)| \le \|f\|_2 \cdot \|g\|_2$ from which assertion 1) of Theorem A follows.

2) Assuming in 3) of Theorem 1 $p = 2$ ($\Longrightarrow p' = 2 \Longrightarrow \gamma = p'/2 = 1$) and taking into account that $M(2) = 1$, one obtains the estimation $\rho_n^{(1)}(h) \le E_n(f)_2 \cdot E_n(g)_2$. The last estimation is exact on the class of functions $h \in A(T)$ representable in the form of convolution $h = f * g$ of two functions $f \in L_2(T)$ and $g \in L_2(T)$, namely, for any function $h \in A(T)$ there exist the functions $f_0 \in L_2(T)$ and $g_0 \in L_2(T)$ such that $f_0 * g_0 = h$ and $\rho_n^{(1)}(h) = E_n(f_0)_2 \cdot E_n(g_0)_2$. Really, assuming (see the proof of point 2) of Theorem A in the cited reference) $c_n(f_0) = |c_n(h)|^{1/2}$ and $c_n(g_0) = |c_n(h)|^{1/2} \cdot sign\,(c_n(h))$ we have $\|f_0\|_2 = \|g_0\|_2 = \left(\rho_0^{(1)}(h)\right)^{1/2} < \infty$; consequently, f_0, $g_0 \in L_2(T)$, $f_0 * g_0 = h$ and

$$E_n(f_0)_2 \cdot E_n(g_0)_2 = \|f_0 - S_n(f_0)\|_2 \cdot \|g_0 - S_n(g_0)\|_2 =$$

$$= \left(\rho_n^{(1)}(h)\right)^{1/2} \cdot \left(\rho_n^{(1)}(h)\right)^{1/2} = \rho_n^{(1)}(h).$$

Further, we need the known Hardy-Littlewood theorem (see for example, [4], §10.3, p.657-658; [1], v.2, §12.6, lemma (6.6) in p.193; [2], v.1, §7.3.5, p.148-149).

Theorem C. *Let* $1 < p < \infty$, $\varphi(x) = \sum_{n=1}^{\infty} c_n(\varphi)\, e^{inx}$, *where* $0 < c_n(\varphi) \downarrow 0$ $(n \uparrow \infty)$; *then* $\varphi \in L_p(T) \Longleftrightarrow \sum_{n=1}^{\infty} n^{p-2} c_n^p(\varphi) < \infty$, *by that*

$$\|\varphi\|_p \asymp \left(\sum_{n=1}^{\infty} n^{p-2} c_n^p(\varphi)\right)^{1/p}.$$

It is shown in the following theorem that estimation 3) of Theorem 1 is exact in the sense of order in the scale of power majorants of sequences of the best approximations of functions f, $g \in L_p(T)$, where $1 < p \le 2$.

Theorem 2. *Let* $1 < p \le 2$, $\gamma = p'/2 \ge 1$, $0 < \alpha \in R$, $0 < \beta \in R$; *there exist the functions* $f_{\alpha,p}(\cdot) \in L_p(T)$, $g_{\beta,p}(\cdot) \in L_p(T)$ *with* $E_n(f_{\alpha,p})_p \underset{(a,p)}{\asymp}$

$n^{-\alpha}$, $E_n(g_{\beta,p})_p \underset{(\beta,p)}{\asymp} n^{-\beta}$, $n \in N$, *such that* [1] $\rho_n^{(\gamma)}(f_{\alpha,p} * g_{\beta,p}) \underset{(\alpha,\beta,p)}{\asymp} n^{-(\alpha+\beta)}$, $n \in N$.

Proof. Put $(p' = p/(p-1))$

$$c_n(f_{\alpha,p}) = \left\{n^{-(\alpha+1/p')}\text{for } n \geq 1;\ 0 \text{ for } n \leq 0\right\},$$

$$c_n(g_{\beta,p}) = \left\{n^{-(\beta+1/p')}\text{for } n \geq 1; 0 \text{ for } n \leq 0\right\}.$$

Since $c_n(f_{\alpha,p}) \downarrow 0\ (n \uparrow \infty)$, $c_n(g_{\beta,p}) \downarrow 0\,(n \uparrow \infty)$ and

$$\sum_{n=1}^{\infty} n^{p-2} c_n^p(f_{\alpha,p}) = \sum_{n=1}^{\infty} n^{p-2} n^{-p(\alpha+1/p')} = \sum_{n=1}^{\infty} n^{-(p\alpha+1)} < \infty,$$

$$\sum_{n=1}^{\infty} n^{p-2} c_n^p(g_{\beta,p}) = \sum_{n=1}^{\infty} n^{p-2} n^{-p(\beta+1/p')} = \sum_{n=1}^{\infty} n^{-(p\beta+1)} < \infty,$$

then we have $f_{\alpha,p} \in L_p(T)$, $g_{\beta,p} \in L_p(T)$ by Theorem C.

Further, by obvious inequality $E_n(\varphi)_p \leq \|\varphi - S_n(\varphi)\|_p$ and Riesz inequality (6), we obtain

$$E_n(f_{\alpha,p})_p \underset{(p)}{\asymp} \|f_{\alpha,p} - S_n(f_{\alpha,p})\|_p \underset{(p)}{\asymp} \left(\sum_{\nu=n+1}^{\infty} \nu^{p-2} c_\nu^p(f_{\alpha,p})\right)^{1/p} =$$

$$= \left(\sum_{\nu=n+1}^{\infty} \nu^{-(p\alpha+1)}\right)^{1/p} \underset{(\alpha,p)}{\asymp} n^{-\alpha},\ n \in N.$$

It is analogously proved that $E_n(g_{\beta,p})_p \underset{(\beta,p)}{\asymp} n^{-\beta}$, $n \in N$. Besides, we see that $f_{\alpha,p} \in A(T)$, $g_{\beta,p} \in A(T)$ for $1/p < \alpha,\ \beta < \infty$ and $f_{\alpha,p} \notin A(T)$, $g_{\beta,p} \notin A(T)$ for $0 < \alpha,\ \beta \leq 1/p$. Finally, by equality (3) we have $(\gamma = p'/2)$

$$\rho_n^{(\gamma)}(f_{\alpha,p} * g_{\beta,p}) = \left(\sum_{\nu=n+1}^{\infty} |c_\nu(f_{\alpha,p}) \cdot c_\nu(g_{\beta,p})|^\gamma\right)^{1/\gamma} =$$

$$= \left(\sum_{\nu=n+1}^{\infty} \nu^{-(\alpha+1/p')\gamma} \nu^{-(\beta+1/p')\gamma}\right)^{1/\gamma} = \left(\sum_{\nu=n+1}^{\infty} \nu^{-\alpha\gamma-1/2} \nu^{-\beta\gamma-1/2}\right)^{1/\gamma} =$$

$$= \left(\sum_{\nu=n+1}^{\infty} \nu^{-(\alpha+\beta)\gamma-1}\right)^{1/\gamma} \underset{(\alpha,\beta,\gamma)}{\asymp} n^{-(\alpha+\beta)},\quad n \in N.$$

Theorem 2 is proved.

Example 1. Put $f_0(x;1) = \operatorname{Re}\sum_{n=2}^{\infty} (\ln n)^{-1} e^{inx} = \sum_{n=2}^{\infty} (\ln n)^{-1} \cos nx$. Since the sequence $a_n = (\ln n)^{-1}$ is convex and $a_n \downarrow 0\ (n \uparrow \infty)$, then by theorem 4 [4; §1.30, p.100] the series in the right hand side converges everywhere, except the points $x \equiv 0 \pmod{2\pi}$, to a nonnegative summable function and is its Fourier series, so that $f_0 \in L_1(T)$ and $c_n(f_0) = a_n = (\ln n)^{-1}$, $n \geq 2$. Further, put

[1] $y_n \underset{(\alpha,\beta,p)}{\asymp} z_n$ means the existence of such constants $0 < c_1 \leq c_2$ depending only on mentioned parameters α, β and p, that $c_1 z_n \leq y_n \leq c_2 z_n$.

$g_0(x;1) = \operatorname{Im}\sum_{n=1}^{\infty} n^{-\alpha}e^{inx} = \sum_{n=1}^{\infty} n^{-\alpha}\sin nx$, where $0 < \alpha \le 1$. Since the sequence $b_n = n^{-\alpha} \downarrow 0$ $(n \uparrow \infty)$ and $\sum_{n=1}^{\infty} n^{-1}b_n < \infty$, then by theorem 7.3.3 [2; v.1, §7.3, p.146] (see also [4; §10.2, p.650-652]) the series in the right hand side converges everywhere, is a Fourier series of the function g_0, and consequently, $c_n(g_0) = b_n = n^{-\alpha}$, $n \ge 1$, and $g_0 \in L_1(T)$. The convolution of these functions (see formula (1.9) in p.65 in [1], v.1, §2.1) $h_0(x) = f_0(x) * g_0(x) = \sum_{n=2}^{\infty} a_n b_n \sin nx$ does not belong to the class $A(T)$, since $\rho_0^{(1)}(h_0) = \sum_{n=2}^{\infty} (n^{\alpha}\ln n)^{-1} = +\infty$ for $0 < \alpha \le 1$. We also note that $f_0 \notin A(T)$ and $g_0 \notin A(T)$.

Example 2. Put $(1 < p < 2 \Longrightarrow p' = p/(p-1) > 2)$ $c_n(f_0) = c_n(g_0) = = \left\{ \left(n^{1/p'}\ln n\right)^{-1} \ \ for \ \ n \ge 2; \ \ 0 \ \ for \ n \le 1 \right\}$.

Since $c_n(f_0) \downarrow 0$ $(n \uparrow \infty)$ and $(p > 1)$

$$\sum_{n=2}^{\infty} n^{p-2}c_n^p(f_0) = \sum_{n=2}^{\infty} n^{p-2}n^{-p/p'}(\ln n)^{-p} = \sum_{n=2}^{\infty} n^{-1}(\ln n)^{-p} < \infty,$$

then, by Theorem C, we have $f_0 \in L_p(T)$ as well $g_0 \in L_p(T)$. The convolution of these functions $h_0 = f_0 * g_0 \notin A(T)$, since $\rho_0(h_0) = \sum_{n=2}^{\infty} |\, c_n(h_0)| = = \sum_{n=2}^{\infty} n^{-2/p'}(\ln n)^{-2} = +\infty$ by virtue of the fact that $2/p' < 1$. We also note that $f_0, g_0 \notin A(T)$.

Example 3. Let $1 < p \le 2$; put (see Example 2)

$$c_n(f_0) = c_n(g_0) = \left\{ \left(n^{1/p'}\ln n\right)^{-1} \quad for \ n \ge 2; \quad 0 \quad for \ n \le 1 \right\}.$$

Then $f_0, g_0 \in L_p(T)$ since in the case $p = 2$ we have $(p = 2 \Longrightarrow p' = 2)$

$$\|f_0\|_2 = \|g_0\|_2 = \left(\sum_{n=2}^{\infty} |\, c_n(f_0)|^2\right)^{1/2} = \left(\sum_{n=2}^{\infty} n^{-1}(\ln n)^{-2}\right)^{1/2} < +\infty.$$

Further, for any number $\beta < p'/2$, we obtain

$$\sum_{n=2}^{\infty} |\, c_n(h_0)|^{\beta} = \sum_{n=2}^{\infty} |\, c_n(f_0)\cdot c_n(g_0)|^{\beta} = \sum_{n=2}^{\infty} n^{-2\beta/p'}(\ln n)^{-2\beta} = +\infty$$

since $2\beta/p' < 1$ and consequently, $h_0 = f_0 * g_0 \notin A^{(\beta)}(T)$.

Theorem 3. *Let* $2 < p \le \infty$, $f \in L_p(T)$, $g \in L_p(T)$, $h = f * g$; *then* $h \in C(T)$ *and the following estimations are valid*

1) $\|h\|_{\infty} \le \|f\|_p \cdot \|g\|_p$;
2) $\sum_{n\in Z} |\, c_n(h)| \le \|f\|_p \cdot \|g\|_p$;
3) $\rho_n^{(1)}(h) = \sum_{|\nu|=n+1}^{\infty} |\, c_{\nu}(h)| \le E_n(f)_p \cdot E_n(g)_p$, $n \in Z_+$.

Proof. 1) The statement $h \in C(T)$ and estimation 1) hold according to point 1) of Remark 1 and inequality in 1) of Theorem 1 in the case $p = 2$ $(\Longrightarrow \gamma' = \infty)$:

$\|h\|_\infty \leq \|f\|_2 \cdot \|g\|_2 \leq \|f\|_p \cdot \|g\|_p$. Further, assuming $p = 2$ in inequalities 2) and 3) of Theorem 1 (see also Remark 1), we obtain ($p = 2 \Longrightarrow p' = 2 \Longrightarrow \gamma = p'/2 = 1$)

$$\sum_{n \in Z} \mid c_n(h) \mid \ \leq \|f\|_2 \cdot \|g\|_2 \leq \|f\|_p \cdot \|g\|_p \,,$$

$$\rho_n^{(1)}(h) \leq E_n(f)_2 \cdot E_n(g)_2 \leq E_n(f)_p \cdot E_n(g)_p \,,$$

and we are done.

Estimation 3) of Theorem 3 is exact in the sense of order in the scale of power majorants of sequences of the best approximations of functions $f \in L_p(T)$ and $g \in L_p(T)$, $2 < p \leq \infty$, namely the following one is valid

Theorem 4. *Let* $2 < p \leq \infty, 0 < \alpha \in R, 0 < \beta \in R.$ *There exist the functions* $f_\alpha(\cdot) \in C(T) \subset L_p(T)$, $g_\beta(\cdot) \in C(T) \subset L_p(T)$ *with* $E_n(f_\alpha)_\infty = O(n^{-\alpha})$, $E_n(g_\beta)_\infty = O\left(n^{-\beta}\right)$, $n \in N$, *such that* $\rho_n^{(1)}(f_\alpha * g_\beta) \underset{(\alpha,\beta)}{\asymp} n^{-(\alpha+\beta)}$, $n \in N$.

Proof. Put (see [1], v.1, §5.4, p.317)

$$f_\alpha(x) = \sum_{n=1}^{\infty} e^{in \ln n} n^{-(\alpha+1/2)} e^{inx}, \ g_\beta(x) = \sum_{n=1}^{\infty} e^{in \ln n} n^{-(\beta+1/2)} e^{inx}.$$

By virtue of the fact that partial sums of series on the right hand side converge uniformly on T (see there p.320) for any $\alpha > 0$ and $\beta > 0$, then sums of these series $f_\alpha \in C(T)$ and $g_\beta \in C(T)$. These series are Fourier series of their sums, so that the corresponding equalities hold, moreover $c_n(f_\alpha) = e^{in \ln n} n^{-(\alpha+1/2)}$ and $c_n(g_\beta) = e^{in \ln n} n^{-(\beta+1/2)}$, $n \in N$. Further, taking into account the estimation $\left|\sum_{\nu=1}^{m} e^{i\nu \ln \nu} e^{i\nu x}\right| \leq c_0 \cdot m^{1/2}$, $m \in N$, uniform with respect to $x \in T$, where c_0 is an absolute constant (see there p.319; theorem (4.7)), and applying the Abel transformation, we obtain

$$f_\alpha(x) = \sum_{n=1}^{\infty} \left(n^{-(\alpha+1/2)} - (n+1)^{-(\alpha+1/2)}\right) \sum_{\nu=1}^{n} e^{i\nu \ln \nu} e^{i\nu x},$$

$$g_\beta(x) = \sum_{n=1}^{\infty} \left(n^{-(\beta+1/2)} - (n+1)^{-(\beta+1/2)}\right) \sum_{\nu=1}^{n} e^{i\nu \ln \nu} e^{i\nu x}.$$

Let us estimate $E_n(f_\alpha)_\infty$ and $E_n(g_\beta)_\infty$. We have

$$E_n(f_\alpha)_\infty \leq \left\| \sum_{m=n+1}^{\infty} \left(m^{-(\alpha+1/2)} - (m+1)^{-(\alpha+1/2)}\right) \sum_{\nu=1}^{m} e^{i\nu \ln \nu} e^{i\nu x} \right\|_\infty \leq$$

$$\leq \sum_{m=n+1}^{\infty} \left(m^{-(\alpha+1/2)} - (m+1)^{-(\alpha+1/2)}\right) \left\| \sum_{\nu=1}^{m} e^{i\nu \ln \nu} e^{i\nu x} \right\|_\infty \leq$$

$$\leq \sum_{m=n+1}^{\infty} (\alpha + 1/2) \ m^{-(\alpha+1/2+1)} \cdot c_0 \cdot m^{1/2} =$$

$$= c_0 \cdot (\alpha + 1/2) \sum_{m=n+1}^{\infty} m^{-(\alpha+1)} \leq c_1(\alpha) \cdot n^{-\alpha},$$

where $c_1(\alpha) = c_0 \cdot (\alpha + 1/2) \cdot \alpha^{-1}$, whence $E_n(f_\alpha)_\infty \le c_1(\alpha)\ n^{-\alpha},\ n \in N$. Analogously we obtain the estimation $E_n(g_\beta)_\infty \le c_1(\beta)\ n^{-\beta},\ n \in N$.

Further, we note that $f_\alpha \in A(T)$ for $\alpha \in (1/2, \infty)$, $f_\alpha \notin A(T)$ for $\alpha \in (0, 1/2]$, and $g_\beta \in A(T)$ for $\beta \in (1/2, \infty)$, $g_\beta \notin A(T)$ for $\beta \in (0, 1/2]$, however the convolution of these functions $f_\alpha * g_\beta \in A(T)$ for any α, $\beta \in (0, \infty)$ since by virtue of formula (3)

$$\rho_0^{(1)}(f_\alpha * g_\beta) = \sum_{n=1}^{\infty} |\ c_n(f_\alpha) \cdot c_n(g_\beta)| = \sum_{n=1}^{\infty} n^{-(\alpha+\beta+1)} < \infty$$

and

$$\rho_n^{(1)}(f_\alpha * g_\beta) = \sum_{\nu=n+1}^{\infty} \nu^{-(\alpha+\beta+1)} \underset{(\alpha,\beta)}{\asymp} n^{-\ (\alpha+\beta)},\ n \in N.$$

Theorem 4 is proved.

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