Aygun T. HUSEYNOVA

ASYMPTOTICS OF SOLUTION OF AN INITIAL VALUE PROBLEM FOR A SYSTEM OF SINGULARLY PERTURBED LINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF FREDHOLM TYPE

Abstract

In the proposed paper it is constructed and researched the asymptotics of solution of initial value problem for the system of singular perturbed liner integro-differential equations of Fredholm's type and it is shown that integral term plays main role for boundedness of solution of the constructed problem as $\mu \to 0$.

It is known that for asymptotic stability of the linear system

$$\frac{dz}{dt} = Az + B(t), \quad z(0) = z_0, \quad 0 \le t \le T$$

it is necessary and sufficient that real parts of all characteristic numbers of a matrix A be negative. This condition is of great importance while investigating asymptotic behaviour of solution of the singular perturbed system

$$\mu \frac{dz}{dt} = Az + B(t), \qquad z(0,\mu) = z_0, \quad 0 \le t \le T \ ,$$

where μ is a small parameter.

By fulfilling the stability condition the investigation of the asymptotics of solution of the problem with initial conditions for singularly perturbed integro-differential equations is completely similar to the investigation of asymptotics of solution of an initial value-problem for differential equations considered for example in [1-4], [8-12]. Asymptotic expansions of solutions of these two problems have the same form. The difference here is in that equations for the coefficients of asymptotic expansion in the case of integro-differential equations are more complicated than in the case of differential equations.

Thus, we can say that the presence of an integral term in the equation by fulfilling the stability condition leads only to some complication of the algorithm for constructing asymptotic expansion, but qualitative behaviour of the solution doesn't change. In the given paper the asymptotics of solution of Cauchy problem is studied in systems of singularity perturbed linear integro-differential equations without stability condition. It is shown that he presence of an integral term leads to qualitative change of the behaviour of solution.

1. Problem statement.

Let's consider the system of linear integro-differential equations with small parameter at the derivative

$$\mu \frac{dz}{dt} = A(t) z(t,\mu) + \int_0^T K(t,s) z(s,\mu) ds + B(t) \ , \tag{1}$$

with initial condition:

$$z(0,\mu) = z_0, \tag{2}$$

where $0 \leq t \leq T$, $0 \leq s \leq T$, $A(t)$ and $K(t,s)$ are $N \times N$ dimensional matrices, $B(t)$ is N dimensional vector column, $\mu > 0$ is a small parameter. We'll assume that the following conditions are fulfilled:

a) $A(t), B(t), K(t)$ have continuous derivatives of m-th order, where m is a sufficiently large natural number, moreover $K(t,s) \not\equiv 0$;

b) characteristic exponents $\lambda_i(t)$ of the matrix $A(t)$ satisfy the condition

$$\operatorname{Re} \lambda_i(t) > 0, \quad i = \overline{1,N}, \quad t \in [0,T] \ . \tag{3}$$

Note that by fulfilling condition (3) the problem

$$\mu \frac{dz}{dt} = A(t) z(t,\mu) + B(t), \quad 0 \leq t \leq T,$$

$$z(0,\mu) = z_0$$

generally speaking, has no bounded solution as $\mu \to 0$,where as problem (1)-(3) has a bounded solution $z(t,\mu)$ as $\mu \to 0$ and its limit is the solution of the system of integral equations

$$0 = A(t) \bar{z} + \int_0^T K(t,s) \bar{z}(s) ds + \tilde{B}(t),$$

where $\tilde{B}(t)$ is determined by the data of the problem.

We call this problem the main problem.

2. Auxiliary problem and asymptotics of its solution.

Let's consider system (1) with the following condition

$$z(T,\mu) = d(\mu)/\mu. \tag{4}$$

Under

$$d(\mu) = \begin{pmatrix} d^{(1)} \\ \vdots \\ d^{(N)} \end{pmatrix}$$

we understand for now unknown N-dimensional vector-column whose each component $d^{(i)}(\mu)$, $i = \overline{1,N}$ is represented in the form of asymptotic series by degrees μ as $\mu \to 0$

$$d^{(i)}(\mu) = d_{-1}^{(i)} + \mu d_0^{(i)} + ... + \mu^{k+1} d_k^{(i)} + ... \qquad i = \overline{1,N}, \tag{5}$$

where $d_k^{(i)}$ are the components of $d_k - N$-dimensional vector-column, $k = -1, 0, 1, ...$, $i = \overline{1,N}$. Then $d(\mu)$ is representable in the form of asymptotic series by degrees μ as $\mu \to 0$:

$$d(\mu) = d_{-1} + \mu d_0 + \mu^2 d_1 + ... + \mu^{k+1} d_k + ... \tag{6}$$

Granting the denotation mentioned above, we can write the components of the vector column $z(T,\mu)$ as

$$z_i(T,\mu) = d^{(i)}(\mu)/\mu, \qquad i = \overline{1,N} \tag{7}$$

We'll search the solution of problem (1), (4) under conditions a)-b) (see for example, [1], [4-8]) in the form of series:

$$z(t,\mu) = \bar{z}_0(t) + \mu \bar{z}_1(t) + ... + \mu^k \bar{z}_k(t) + ... + \tfrac{1}{\mu}\Pi_{-1} z(\tau) + \Pi_0 z(\tau) + \\ + \mu \Pi_1 z(\tau) + ... + \mu^k \Pi_k z(\tau) + ... , \tag{8}$$

where $\tau = \dfrac{t-T}{\mu}$.

Therefore, the components of the vector

$$z(t,\mu) = \begin{pmatrix} z_1(t,\mu) \\ \vdots \\ z_N(t,\mu) \end{pmatrix}$$

are represented as:

$$z_i(t,\mu) = \bar{z}_0^{(i)}(t) + \mu \bar{z}_1^{(i)}(t) + ... + \mu^k \bar{z}_k^{(i)}(t) + ... + \tfrac{1}{\mu}\Pi_{-1} z^{(i)}(\tau) + \Pi_0 z^{(i)}(\tau) +$$

$$+\mu \Pi_1 z^{(i)}(\tau) + ... + \mu^k \Pi_k z^{(i)}(\tau) + ... , \qquad i = \overline{1,N}\ . \tag{9}$$

Here $\bar{z}_k^{(i)}(t)$ and $\Pi_{k-1} z^{(i)}(\tau)$ are the components of the vectors $\bar{z}_k(t)$ and $\Pi_{k-1} z(\tau)$, $i = \overline{1,N}$, $k = 0, 1, 2...$ respectively. Putting (8) in (1) and taking all mentioned above denotation into account, we get

$$\mu\tfrac{d}{dt}\left(\bar{z}_0(t)+...+\mu^k\bar{z}_k(t)+...\right)+$$

$$+\tfrac{d}{d\tau}\left(\tfrac{1}{\mu}\Pi_{-1}z(\tau)+\Pi_0 z(\tau)+...+\mu^k\Pi_k z(\tau)+...\right)=$$

$$=A(t)\left(\bar{z}_0(t)+...+\mu^k\bar{z}_k(t)+...\right)+A(\tau\mu+T)\times$$

$$\times\left(\tfrac{1}{\mu}\Pi_{-1}z(\tau)+\Pi_0 z(\tau)+...+\mu^k\Pi_k z(\tau)+...\right)+ \tag{10}$$

$$+\int_0^T K(t,s)\left(\bar{z}_0(s)+...+\mu^k\bar{z}_k(s)+...\right)ds+$$

$$+\mu\int_{-\infty}^0 K(t,\sigma\mu+T)\left(\frac{1}{\mu}\Pi_{-1}z(\sigma)+...+\mu^k\Pi_k z(\sigma)+...\right)d\sigma+B(t),$$

where $\sigma=\dfrac{s-T}{\mu}$.

Representing the right hand side of (10) in the form of series by the degrees μ and comparing the coefficients at the same degrees μ in the both parts of equality (10) (depending on τ and depending on t separately) we get the equation for defining $\Pi_{k-1}z(\tau),\ \bar{z}_k(t)\quad(k=0,1,2,...)$. Similarly, putting series (8) into (4), substituting $d(\mu)$ by expression (6) and equalling the coefficients at the same degrees of μ, we get an initial condition for the terms of series (8).

For $\Pi_{-1}z(\tau)$ we get the system of differential equations

$$\frac{d\Pi_{-1}z}{d\tau}=A(T)\Pi_{-1}z$$

with initial condition

$$\Pi_{-1}z(0)=d_{-1},$$

whence we get

$$\Pi_{-1}z(\tau)=\exp(A(T)\tau)d_{-1}\quad(\tau\le 0), \tag{11}$$

where $A(T)$ is a constant $N\times N$-dimensional matrix, and $\exp(A(T)\tau)$ is a matrix exponent.

With respect to $\bar{z}_0(t)$ we get the system of integral equations of the form:

$$0=A(t)\bar{z}_0(t)+\int_0^T K(t,s)\bar{z}_0(s)ds+\int_{-\infty}^0 K(t,T)\Pi_{-1}z(\sigma)d\sigma+B(t)\ .$$

Put here expression (11) for $\Pi_{-1}z(\tau)$, write the last equation in the form

$$\bar{z}_0(t) = \int_0^T \bar{K}(t,s)\,\bar{z}_0(s)\,ds + f_0(t) \ , \tag{12}$$

where

$$\bar{K}(t,s) = -A^{-1}(t)\,K(t,s)\,, \ \ f_0(t) = \left[A^{-1}(T)\,\bar{K}(t,T)\right]d_{-1} - A^{-1}(t)\,B(t) \ .$$

Let 1 be a regular value of the kernel $\bar{K}(t,T)$, and let $R(t,s)$ be its resolvent. The solution of equation (12) write in the form

$$\bar{z}_0(t) = f_0(t) + \int_0^T \bar{R}(t,s)\,f_0(s)\,ds \ .$$

Put here expression for $f_0(t)$:

$$\bar{z}_0(t) = A^{-1}(T)\left[\bar{K}(t,T) + \int_0^T \bar{R}(t,s)\,\bar{K}(s,t)\,ds\right]d_{-1} +$$
$$+\left[-A^{-1}(t)\,B(s)\,\bar{R}(t,s)\,ds\right] \ .$$

Since for $\bar{R}(t,s)$ it is valid the equality

$$\bar{R}(t,s) = \bar{K}(t,s) + \int_0^T \bar{R}(t,p)\,\bar{K}(p,s)\,dp \ ,$$

then the coefficients for d_{-1} is equal to $A^{-1}(T)\,\bar{R}(t,T)$. Introduce the denotation

$$\tilde{z}_0(t) = -A^{-1}(t)\,B(t) - \int_0^T A^{-1}(s)\,B(s)\,\bar{R}(t,s)\,ds \ .$$

Then

$$\bar{z}_0(t) = A^{-1}(T)\,\bar{R}(t,T)\,d_{-1} + \tilde{z}_0(t) \ . \tag{13}$$

As is seen from (13), $\bar{z}_0(t)$ is linearly dependent on d_{-1}. We use this linear dependence in further consideration of the problem (1), (2), (3).

Similarly we can define $\Pi_0 z(\tau)\,,\ \bar{z}_1(t)\,,\ \Pi_1 z(\tau)\,,\ \bar{z}_2(t)\,,\ldots$. To define $\Pi_k z(\tau)$ at each k $(k = 0,1,2,\ldots)$ we get the differential equation

$$\frac{d\Pi_k z(\tau)}{d\tau} = A(T)\,\Pi_k z(\tau) + \exp\left[A(T)\,\tau\right]p_k(\tau) \ ,$$

where $p_k(\tau)$ is explicitly expressed by the data of the problem, with initial condition

$$\Pi_k z(0) = d_k - \bar{z}_k(T) \ .$$

Hence we get

$$\Pi_k z(\tau) = \exp(A(T)\tau)[d_k - \bar{z}_k(T)] + \exp[A(T)\tau]\left[\int_0^\tau p_k(\sigma)\,d\sigma\right]. \quad (14)$$

To determine the functions $\bar{z}_k(t)$ at each k $(k = 1, 2, ...)$ we get the integral equation similar to (12)

$$\bar{z}_k(t) = \int_0^T \bar{K}(t,s)\,\bar{z}_k(s)\,ds + f_k(t), \quad k = 1, 2, ..., \quad (15)$$

where $f_k(t)$ in a certain way is expressed by the data of the problem.

Acting as in the case of defining of $\bar{z}_0(t)$ we get

$$\bar{z}_k(t) = A^{-1}(T)\,\bar{R}(t,T)\,d_{k-1} + \tilde{z}_k(t), \quad (16)$$

where $\tilde{z}_k(t)$ is some vector-function.

The linear dependence of $\tilde{z}_k(t)$ on d_{k-1} will be also used below by considering the main problem. By $z_n(t,\mu)$ denote a partial sum of series (8) of order n:

$$z_n(t,\mu) = \frac{1}{\mu}\Pi_{-1}z(\tau) + \sum_{k=0}^{n}\mu^k[\bar{z}_k(t) + \Pi_k z(\tau)]. \quad (17)$$

Theorem 1. *Let*

1) The conditions a)-b) be fulfilled;

2) 1 be a regular value of the kernel $\bar{K}(t,s)$.

Then there will be found such numbers $\mu_0 > 0$ *and* $c > 0$ *that for* $0 < \mu \le \mu_0$ *the solution* $z(t,\mu)$ *of auxiliary problem (1), (4) exists, is unique and satisfies the inequality*

$$\|z(t,\mu) - z_n(t,\mu)\| \le c\mu^{n+1} \quad at \quad 0 \le t \le T. \quad (18)$$

Proof. The existence and uniqueness of the solution of the auxiliary problem follow from the linearity of the system of equations (1).

Let's prove inequality (18). To this end we put

$$u(t,\mu) = z(t,\mu) - z_n(t,\mu).$$

With respect to $u(t,\mu)$ we get:

$$\mu\frac{du}{dt} = A(t)\,u + \int_0^T K(t,s)\,u(s,\mu)\,ds + H(t,\mu)\ ,) \quad (19)$$

where

$$H(t,\mu) = A(t)\,z_n(t,\mu) + \int_0^T K(t,s)\,z_n(s,\mu)\,ds + B(t) - \mu\frac{dz_n(t,\mu)}{dt}.$$

Putting in $H(t,\mu)$ expression (17), making substitution $t=\tau\mu+T,\ s=\sigma\mu+T$ in corresponding addends and using the equation for $\Pi_k z(\tau)\quad (k=-1,0,1,...,n)$ and $\bar{z}_k(t)\quad (k=0,1,...,n)$ we can get the estimate

$$\|H(t,\mu)\|\leq c\mu^{n+1}\quad at\quad 0\leq t\leq T,\quad 0<\mu\leq\mu_0\ . \tag{20}$$

Obviously, the initial condition for $u(t,\mu)$ is of the form

$$u(T,\mu)=\left(\mu^{n+2}d_{n+1}+...\right)/\mu\ ,$$

whence

$$\|u(T,\mu)\|\leq c\mu^{n+1}\quad at\quad 0<\mu\leq\mu_0\ . \tag{21}$$

We can write the solution of equation (19) in the form

$$u(t,\mu)=\Phi(t,T,\mu)\,U(t,\mu)+\frac{1}{\mu}\int_0^t\Phi(t,s,\mu)\,H(s,\mu)\,ds\ , \tag{22}$$

where $\Phi(t,s,\mu)$ is the solution of the homogeneous equation

$$\mu\frac{d\Phi(t,s,\mu)}{dt}=A(t)\,\Phi(t,s,\mu)\qquad(0\leq s\leq t\leq T)\ ,$$

satisfying the condition

$$\Phi(s,s,\mu)=E_N\ ,$$

where E_N is a unique $N\times N$-dimensional matrix

$$\|\Phi(t,s,\mu)\|\leq c\exp\left(\frac{-\chi(t-s)}{\mu}\right)\ , \tag{23}$$

where $0\leq s\leq t\leq T\ ,\quad 0<\mu\leq\mu_0,\quad \chi>0.$

From 22 by inequalities (20), (21) and (23) it immediately follows the inequality

$$\|u(t,\mu)\|\leq c\mu^{n+1}\quad at\quad 0\leq t\leq T,\quad 0<\mu\leq\mu_0\ .$$

that proves (18) and thereby the theorem.

3. Asymptotic solution of the main problem.

Now let's go back to the main problem. For its solution let's use asymptotic of solution $z(t,d,\mu)$ of auxilary problem (1), (4), (3). Choose the vector d so that $z(t,d,\mu)$ satisfies condition (2). Then, with respect to d we get the following system

$$z(0,d,\mu)=z_0\ . \tag{24}$$

We shall search solution (24) in the form

$$d(\mu) = d_{-1}\mu d_0 + \mu^2 d_1 + ... + \mu^{k+1} d_k + ...$$

In order to find vector-coefficients $d_{-1}, d_0, d_1, ...$ into (24) we put instead of the exact solution $z(0, d, \mu)$ its asymptotic expansion (8). Since at the point $t = 0$ all frontier vector-functions have the estimate

$$\|\Pi_k z(-T/\mu)\| \leq c \exp(-\chi T/\mu), \quad 0 < \chi \leq \|A(T)\| \ ,$$

(24) takes the form

$$\bar{z}_0(0) + \mu \bar{z}_1(0) + ... + \mu^k \bar{z}_k(0) + ... = z_0 \ .$$

Granting for the dependence of $\bar{z}_k(0)$ on d_{k-1} (13) and (16), with respect to $d_{-1}, d_0, ...$ we get the system of linear equations

$$\bar{z}_0(0) = A^{-1}(T)\bar{R}(0, T) d_{-1} + \tilde{z}_0(0) = z_0, \tag{25}$$

$$\bar{z}_k(0) = A^{-1}(T)\bar{R}(0, T) d_{k-1} + \tilde{z}_k(0) = 0, \quad (k = 1, 2, ...) \ . \tag{26}$$

Let there exists $\bar{R}^{-1}(0, T)$. Then equations (25) and (26) are uniquely solvable with respect to d_{-1}, d_{k-1}. Denote these solutions $\bar{d}_k$, $k = -1, 0, 1, ...$ and construct $z_n(t, \mu)$ defined by formula (17) for $d_{-1} = \bar{d}_{-1}$, $d_0 = \bar{d}_0, ..., d_n = \bar{d}_n$.

Then the following theorem is valid.

Theorem 2. *Assume that the conditions of theorem 1 are fulfilled and exists $\bar{R}^{-1}(0, T)$, where $\bar{R}(t, s)$ is the resolvent of the kernel $\bar{K}(t, s)$, and $\bar{R}(0, T)$ is the value of the resolvent for $t = 0$, $s = T$. Then there will be found such numbers $\mu_0 > 0$ and $c > 0$ that for $0 < \mu \leq \mu_0$ there exists a unique solution $z(t, \mu)$ of problem (1), (2) and it holds the inequality*

$$\|z(t, \mu) - z_n(t, \mu)\| \leq c\mu^{n+1} \quad at \quad 0 \leq t \leq T \ . \tag{27}$$

Proof. The existence and uniqueness of the solution $z(t, \mu)$ of problem (1), (2) follow from the stated above.

Now prove inequality (27). Consider the solution $z(t, d, \mu)$ for

$$d = \left(\bar{d}\right)_n \equiv \bar{d}_{-1} + \mu \bar{d}_0 + ... + \mu^{n+1} \bar{d}_n \ .$$

For it at $t = 0$ by equations (25), (26) where $k = 1, 2, ..., n + 1$, we have

$$z\left(0, \left(\bar{d}\right)_n, \mu\right) = \bar{z}_0(0) + \mu \bar{z}_1(0) + ... + \mu^{n+1} \bar{z}_{n+1}(0) + o\left(\mu^{n+2}\right) = z_0 + o\left(\mu^{n+2}\right) \ .$$

Hence the inequality

$$\left\|\bar{d}(\mu) - \left(\bar{d}\right)_n\right\| \leq c\mu^{n+2} \ ,$$

follows, i.e. for $\bar{d}\,(\mu)$ the asymptotic representation

$$\bar{d}\,(\mu) = \bar{d}_{-1} + \mu\bar{d}_0 + ... + \mu^{n+1}\bar{d}_n + ... + o\left(\mu^{n+2}\right)$$

is valid.

Inequality (27) immediately follows from theorem 1. Theorem 2 is proved.

References

[1]. Butuzov V.F. *To the problem on the asymptotics of solutions of intergro-differntial equations with small parameter at a derivative.* Diff. uravneniya, 1966, v.2, No3, pp.391-406. (Russian)

[2]. Butuzov V.f. *Asymptotics of solution of one problem for nonlinear integro-differential equation with small parameter at a derivative.* Diff. uravneniya, 1968, v.4, No3, pp.491-507. (Russian)

[3]. Vasilyeva A.B. *Asymptotics of solutions of some problems for ordinary non-linear differential equations with small parameter at higher derivative.* Uspekhi math. nauk, 1963, v.18, No3, pp.15-86. (Russian)

[4]. Vasilyeva A.B., Butuzov V.F. *Asymptotical methods in theory of singular perturbations.* Moscow, Visshaya shkola, 1990, 208 p. (Russian)

[5]. Huseynova A.T. *On some properties of asymptotic behaviour of solutions of singularly perturbed integro-differential equations.* Proceedings of the conference devoted to the 80-th anniversary of Akhmedov K.T., BSU, 1998, pp.73-76. (Russian)

[6]. Huseynova A.T. *On the asymptotics of solution of a system of Volterra type singular perturbed integro-differential equations.* Proceedings of Institute of Mathematics and Mechanics, Baku, 2000, v.XII(XX), pp.51-57.

[7]. Il'in A.M. *Boundary layer.* Itogi nauki I tekhniki. Moscow, VINITI, 1988, v.34, pp.175-213. (Russian)

[8]. Imanaliyev M.I. *Oscillation and stability of solutions of singularly perturbed integro-differential systems.* Frunze, Ilim, 1974, 352 p. (Russian)

[9]. Pontryagin L.S. *Asymptotic behaviour of solutions of a system of differential equations with small parameter at higher derivatives.* Izv. ANSSSR, ser. math., 1957, 21, No5, pp.605-626. (Russian)

[10]. Tikhonov A.N. *On dependence of solutions of the differential equation on small parameter.* Mat. sbornik, 1948, v.22 (64), No2, pp.193-204. (Russian)

[11]. Tikhonov A.N. *On systems of differential equations containing parameters.* Mat. sbornik, 1950, v.27 (69), No1, pp.147-156. (Russian)

[12]. Tikhonov A.N. *Systems of differential equations, containing small parameters under derivatives.* Mat. sbornik, 1952, v.31 (73), No3, pp.575-586. (Russian)

Aygun T. Huseynova
Baku State University.
23, Z.I.Khalilov str., AZ1148, Baku, Azerbaijan.
Tel.: (99412) 441 74 07 (off.)

Received October 20, 2003; Revised February 27, 2004.
Translated by Mamedova Sh.N.