Rovshan Z. HUMBATALIYEV

ON m-FOLD COMPLETENESS OF EIGEN AND ADJOINT VECTORS OF A CLASS OF POLYNOMIAL OPERATOR BUNDLES OF A HIGHER ORDER

Abstract

Sufficient conditions are found for m-fold completeness of a part of eigen and adjoint vectors of a class of bundles of higher order corresponding to eigen values from some sector $\widetilde{S_\alpha}$.

Let H be a separable Hilbert space, A be a self-adjoint positive defined operator in H with completely continuous inverse A^{-1}. Let's denote by H_γ a Hilbert scale generated by the operator A. Let $S_\alpha = \{\lambda / |\arg\lambda| < \alpha\}$, $0 < \alpha < \pi/2$ be some sector from a complex plane, and $\widetilde{S_\alpha} = \left\{\lambda / |\arg\lambda - \pi| < \frac{\pi}{2} - \alpha\right\}$. In the given paper we shall search m-fold completeness of eigen and adjoint vectors of the bundle

$$P(\lambda) = \left(-\lambda^2 E + A^2\right)^m + \sum_{j=0}^{2m-1} \lambda^j A_{2m-j}, \tag{1}$$

corresponding to eigen values from the sector $\widetilde{S_\alpha}$. To this end we introduce some notation and denotation. In the sequel, we shall assume the fulfilment of the following conditions: 1) A is a self-adjoint positive defined operator; 2) A^{-1} is a completely continuos operator; 3) The operators $B_j = A_j A^{-j}$, $j = \overline{1, 2m}$ are bounded in H.

Denote by $L_2(R_+ : H)$ a Hilbert space whose elements $u(t)$ are measurable and integrable in the sense of Bohner, i.e.

$$L_2(R_+ : H) = \left\{ u(t) \,/\, \|u(t)\|_{L_2(R_+:H)} = \left(\int_0^\infty \|u(t)\|_H^2 \, dt \right)^{1/2} < \infty \right\}.$$

Let $H_2(\alpha : H)$ be a liner set of holomorphic in $S_\alpha = \{\lambda / |\arg\lambda| < \lambda\}$ vector functions $u(z)$ for which

$$\sup_{|\varphi| < \alpha} \int_0^\infty \left\| u\left(te^{i\varphi}\right) \right\|^2 dt < \infty .$$

The elements of this set have boundary values in the sense of $L_2(R_+ : H)$ and equal to $u_\alpha(t) = u\left(te^{i\alpha}\right)$ and $u_{-\alpha}(t) = u\left(te^{-i\alpha}\right)$. This linear sets turns into Hilbert space with respect to the norm

$$\|u(t)\|_\alpha = \frac{1}{\sqrt{2}}\left(\|u_\alpha(t)\|^2_{L_2(R_+:H)} + \|u_{-\alpha}(t)\|^2_{L_2(R_+:H)}\right)^{1/2}.$$

Denote by $W_2^{2m}(\alpha : H)$ a Hilbert space

$$W_2^{2m}(\alpha : H) = \left\{u(z) / u^{(2m)}(z) \in H_2(\alpha : H),\ A^{2m}u(z) \in H_2(\alpha : H)\right\}$$

with norm

$$|||u|||_\alpha = \left(\left\|u^{(2m)}\right\|^2_\alpha + \left\|A^{2m}u\right\|^2_\alpha\right)^{1/2}.$$

Bind a bundle $P(\lambda)$ form the equality (1) with the following initial value problem:

$$P(d/dz)\,u(z) = 0, \tag{2}$$

$$u^{(j)}(0) = \psi_j, \quad j = \overline{0, m-1}\,, \tag{3}$$

where we'll understand (3) in the sense

$$\lim_{\substack{z\to 0 \\ |\arg z|<\alpha}} \left\|u^{(j)}(z)\right\|_{2m-j-1/2} = 0\,.$$

Definition 1. *If for any $\psi_j \in H_{2m-j-1/2}$ $\left(j = \overline{0, m-1}\right)$ there exists a vector-function $u(z) \in W_2^{2m}(\alpha : H)$ satisfying equation (2) in S_α identically and inequality*

$$|||u|||_\alpha \le const \sum_{j=0}^{m-1} \left\|\psi_j\right\|_{2m-j-1/2}$$

they say that problem (2)-(3) is regularly solvable and $u(z)$ will be called a regular solution of problem (2)-(3).

Let

$$\varphi_0(\lambda) = \left(-\lambda^2 e^{2i\alpha} + 1\right)^4 = \sum_{k=1}^{2m} c_k \lambda^k$$

$$\psi(\lambda, \beta) = \varphi_0(\lambda)\,\varphi_0(-\lambda) - \beta\,(i\lambda)^{2j}, \quad \beta \in [0,_j^{-2})\,.$$

then

$$\psi(\lambda, \beta) = F(\lambda; \beta)\, F\left(-\bar{\lambda}; \beta\right)$$

moreover, $F(\lambda; \beta)$ has roots in the left half-plane and is of the form

$$F(\lambda; \beta) = \sum_{k=1}^{2m} \alpha_k(\beta)\,\lambda^k\,.$$

Denote by $M_{j,m}(\beta)$ a matrix obtained by $M_j(\beta)$ by rejecting m first rows and m first columns, where

$$M_j(\beta)=\frac{1}{2}\left[\tilde{u}_\alpha^{-1}S_j(\beta)\,\tilde{u}_\alpha+\tilde{u}_\alpha S_j(\beta)\,\tilde{u}_\alpha^{-1}\right],$$

$$\tilde{u}_\alpha\equiv diag\left(1,e^{i\alpha},e^{2i\alpha},...,e^{i(m-1)\alpha}\right),\quad \beta\in\left[0,b_j^{-2}\right],$$

$$b_j=\sup_{\xi\in R}\left|\frac{\xi^j}{\left(\xi^4+1+2\xi^2\cos 2\alpha\right)^{m/2}}\right|,\quad \text{and}\quad S_j(\beta)=R_j(\beta)-T,$$

moreover $R_j(\beta)=(r_{pq,j}(\beta))$, $T=(t_{pq})$. For $p>q$

$$r_{pq,j}(\beta)=\sum_{\nu=0}^{\infty}(-1)^\nu\ \overline{\alpha_{p+v,j}(\beta)}\alpha_{q-\nu-1}(\beta)\quad(\alpha_v=0,\ v<0,\ v>2m)$$

$$t_{pq,j}(\beta)=\sum_{\nu=0}^{\infty}(-1)^\nu\ \overline{c_{p+v,j}(\beta)}c_{q-\nu-1}(\beta)\quad(c_v=0,\ v<0,\ v>2m)$$

for $p=q$

$$r_{pq,j}(\beta)=\sum_{\nu=0}^{\infty}(-1)^\nu\ \mathrm{Re}\,\overline{\alpha_{p+v,j}(\beta)}\alpha_{q-\nu-1}(\beta)$$

$$t_{pq,j}(\beta)=\sum_{\nu=0}^{\infty}(-1)^\nu\ \mathrm{Re}\,\overline{c_{p+v,j}(\beta)}c_{q-\nu-1}(\beta),$$

and for $p<q$

$$r_{pq,j}(\beta)=\overline{r_{pq,j}(\beta)},\qquad t_{pq,j}(\beta)=\overline{t_{pq,j}(\beta)}$$

moreover $\alpha_k(\beta)$ and c_k are the coefficients of the polynomial $\varphi_0(\lambda)$ and $F(\lambda;\beta)$. The following theorem is obtained from the results of the works [1] and [2].

Theorem 1. *Let A be a self-adjoint positive defined operator, the operators $b_j=A_jA^{-j}$ $\left(j=\overline{1,2m}\right)$ be bounded in H and the inequality*

$$\sum_{j=1}^{2m}æ_j\left\|B_j\right\|<1,$$

be fulfilled, where the numbers æ are determined as

$$æ_j\geq\begin{cases}b_j, & \text{if}\quad \det M_{j,m}(\beta)\neq 0\\[2ex] \mu_j^{-1/2}, & \text{in otherwise}.\end{cases}\tag{4}$$

Here $\mu_j^{-1/2}$ is the least root of the equation $\det M_{j,m}(\beta)=0$. Then problem (2)-(3) is regularly solvable.

In order to study m-fold completeness of a system of eigen and self-adjoint vectors corresponding to eigen-values from the sector $\widetilde{S_\alpha}$, we have to investigate some analytic properties of the resolvent.

Definition 2. *Let $K\left(\widetilde{S_\alpha}\right)$ be a system of eigen and adjoint vectors corresponding to eigen values from the sector $\widetilde{S_\alpha}$. If for any collection of m vectors from the holomorphy of the vector-function*

$$R(\lambda)=\sum_{j=0}^{2m-1}\left(A^{2m-j-1/2}p^{-1}(\lambda)\right)^* A^{2m-j-1/2}\lambda^j\psi^j$$

in the sector $\widetilde{S_\alpha}$ it yields that $K\left(\widetilde{S_\alpha}\right)$ in strongly m-fold complete in H.

Note that this definition is a prime generalization of m-fold completeness of the system in H in the sense of M.V.Keldysh and in fact it means that the derivatives of the chain in the sense of M.V.Keldysh are complete in the space of traces $\widetilde{H}=\overset{m-1}{\underset{j=0}{\oplus}}H_{m-j-1/2}$.

It holds the following theorem on an analytic property of the resolvent.

Theorem 2. *Let the conditions 1)-3) be fulfilled and it holds the inequality*

$$\sum_{j=0}^{2m-1} b_j\left\|B_{2m-j}\right\|<1\ ,$$

where

$$b_j=\sup_{\xi\in R}\left(\frac{\xi^{2j/m}}{1+\xi^4+2\xi^2\cos 2\alpha}\right)^{m/2},\quad j=\overline{0,2m}\ .$$

Besides, let one of the following conditions be fulfilled:

a) $A^{-1}\in\sigma_p\quad(0<p<\pi/(\pi-2\alpha))$;

b) $A^{-1}\in\sigma_p\quad(0<p<\infty)$, $B_j\ \left(j=\overline{0,2m}\right)$ *are completely continuous operators in H.*

Then the resolvent of the operator bundle $p(\lambda)$ possess the following properties:

1) $A^{2m}p^{-1}(\lambda)$ is represented in the form of relation of two entire fucntions of order not higher than p and has a minimal type order p;

2) there exists a system $\{\Omega\}$ of rays from the sector $\widetilde{S_\alpha}$ where the rays

$$\Gamma_{\frac{\pi}{2}+\alpha}=\left\{\lambda/\arg\lambda=\frac{\pi}{2}+\alpha\right\},\quad \Gamma_{\frac{3\pi}{2}-\alpha}=\left\{\lambda/\arg\lambda=\frac{3\pi}{2}-\alpha\right\},$$

are also contained, and the angle between the neighboring rays is no greater than π/p and the estimation

$$\left\|p^{-1}(\lambda)\right\|\le const\ |\lambda|^{-2m}$$

$$\left\|A^{2m}p^{-1}(\lambda)\right\|\le const$$

holds on these rays.

Proof. Since

$$P(\lambda)A^{-2m}=(E+B_{2m})(E+T(\lambda))\ ,$$

where

$$T(\lambda)=\sum_{j=1}^{2m}c_jA^{-j}\ ,\quad c_j\left(E+B_{2m}\right)^{-1}B_jA^{j-2m},\quad j=1,3,...,2m-1,2m$$

$$c_j=\left(E+B_{2m}\right)^{-1}\left(B_j+(-1)^j\binom{2m}{j}E\right)A^{-2j},\quad j=2,4,...,2m-2\ .$$

Then applying Keldysh [2] lemma, we get that

$$A^{2m}p^{-1}(\lambda)=\left(E+T(\lambda)\right)^{-1}\left(E+B_{2m}\right)^{-1}$$

is represented in the form of relation of two entire functions of order not higher than p and of minimal type at order p.

On the other hand

$$p(\lambda)=p_0(\lambda)+p_1(\lambda)\,,$$

therefore

$$p^{-1}(\lambda)=p_0^{-1}(\lambda)\left(E+p_1(\lambda)p_0^{-1}(\lambda)\right)^{-1}$$

$$A^{2m}p^{-1}(\lambda)=A^{2m}p_0^{-1}(\lambda)\left(E+p_1(\lambda)p_0^{-1}(\lambda)\right)^{-1}\ .$$

Since by fulfilling the condition a) of the theorem we get on the rays $\Gamma_{\frac{\pi}{2}+\alpha}$ and $\Gamma_{\frac{3\pi}{2}-\alpha}$ (i.e. for $\lambda=re^{i\left(\frac{\pi}{2}+\alpha\right)}$, $\lambda=re^{i\left(\frac{3\pi}{2}-\alpha\right)}$)

$$\left\|p_1(\lambda)p_0^{-1}(\lambda)\right\|\le\sum_{j=0}^{2m-1}\left\|B_{2m-j}\right\|\ \left\|\lambda^jA^{2m-j}p_0^{-1}(\lambda)\right\|_{H\to H}\ ,\tag{5}$$

therefore we should first estimate the norm

$$\left\|\lambda^jA^{2m-j}p_0^{-1}(\lambda)\right\|_{H\to H}$$

on the rays $\Gamma_{\frac{\pi}{2}+\alpha}$ and $\Gamma_{\frac{3\pi}{2}-\alpha}$. Let $\lambda=re^{i\left(\frac{\pi}{2}+\alpha\right)}\in\Gamma_{\frac{\pi}{2}+\alpha}$.

Then it follows from the spectral expansion of the operator A

$$\left\|\lambda^jA^{2m-j}p_0^{-1}(\lambda)\right\|_{H\to H}=\sup_{\mu\in\tau(A)}\left|\lambda^j\mu^{2m-j}\left(-\lambda^2+\mu^2\right)^{-m}\right|=$$

$$=\sup_{\mu\in\tau(A)}\left|r^j\mu^{2m-j}\left(r^2e^{2i\alpha}+\mu^2\right)^{-m}\right|=$$

$$=\sup_{\mu\in\tau(A)}\left|r^j\mu^{2m-j}\left(r^4+\mu^4+2r^2\mu^2\cos2\alpha\right)^{-\frac{m}{2}}\right|=$$

$$=\sup_{\mu\in\tau(A)}\left(\frac{r^{\frac{2j}{m}}\mu^{\frac{2(2m-j)}{m}}}{1+\left(\frac{r}{\mu}\right)^4+2\left(\frac{r}{\mu}\right)^2\cos2\alpha}\right)^{\frac{m}{2}}=b_j\ .$$

So, we get from inequality (5)

$$\left\|p_1(\lambda)\,p_0^{-1}(\lambda)\right\| \le \sum_{j=1}^{2m-1}\|B_{2m-j}\|\,b_j < \gamma < 1\ .$$

Therefore on this ray

$$\left\|p^{-1}(\lambda)\right\| \le \left\|p_0^{-1}(\lambda)\right\|\ \left\|\left(E + p_1(\lambda)\,p_0^{-1}(\lambda)\right)^{-1}\right\| \le \left\|p_0^{-1}(\lambda)\right\|\frac{1}{\gamma}\ .$$

On the other hand, on the ray $\Gamma_{\frac{\pi}{2}+\alpha}$ it holds the estimation

$$||p_0^{-1}(\lambda)\,|| = ||\left(-\lambda^2E + A^2\right)^{-m}|| = \sup_{\mu\in\sigma(A)}\left|\frac{1}{\left(\lambda^4+\mu^4+2\lambda^2\mu^2\cos 2\alpha\right)^{\frac{m}{2}}}\right|\ .$$

If $\cos 2\alpha \ge 0\quad (0\le\alpha\le\pi/4)$, then

$$\sup_{\mu\in\sigma(A)}\left|\frac{1}{\left(\lambda^4+\mu^4+2\lambda^2\mu^2\cos 2\alpha\right)^{\frac{m}{2}}}\right| \le$$

$$\le \sup_{\mu\in\sigma(A)}\left|\frac{1}{\left(\lambda^4+\mu^4\right)^{\frac{m}{2}}}\right| \le const\ |\lambda|^{-2m}.$$

It is analogously proved that on these rays

$$\left\|A^{2m}p^{-1}(\lambda)\right\| = const\ .$$

The theorem is proved.

Theorem 3. *Let conditions 1)-3) be fulfilled and it holds*

$$\sum_{j=0}^{2m-1}æ_j\,\|B_{2m-j}\| < 1,$$

where the numbers $æ_j$ are determined from formula (4).

Besides, one of the conditions a) or b) of theorem 2 is fulfilled.

Then the system $K\left(\widetilde{S_\alpha}\right)$ is strongly m-fold complete in H.

Proof. Prove the theorem by contradiction. Then there exists vectors ψ_k $\left(k=\overline{0,m-1}\right)\in H_{2m-k-1/2}$ for which even if one of them differ from zero and

$$R(\lambda) = \sum_{j=0}^{m-1}\left(A^{2m-k-1/2}p^{-1}\left(\bar\lambda\right)\right)^*\lambda^kA^{2m-k-1/2}\psi_k$$

is a holomorphic vector-function in the sector $\widetilde{S_\alpha}$. By theorem 2 and Phragmen-Lindelef theorem in the sector $R(\lambda)$ the $\widetilde{S_\alpha}$ has the estimation

$$\|R(\lambda)\| \le const|\lambda|^{-1/2}\ .$$

On the other hand, by theorem 1 problem (2)-(3) has a unique regular solution $u(z)$ for any $\psi_k \in H_{2m-k-1/2}$. Denote by $\hat{u}(\lambda)$ its Laplace transformation. Then $u(z)$ is represented as

$$u(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{\pi}{2}+1}} \hat{u}(\lambda)\, e^{\lambda z} d\lambda - \frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{3\pi}{2}-\alpha}} \hat{u}(\lambda)\, e^{\lambda z} d\lambda \ ,$$

where $\hat{u}(\lambda) = p^{-1}(\lambda)\, g(\lambda)$ and $g(\lambda) = \sum\limits_{j=0}^{m-1} \lambda^{m-j} Q_j u^{(j)}(0)\ ,$ Q_j are some operators, obviously, for $t > 0$

$$\sum_{k=0}^{m-1} \left(u^{(k)}(t)\, , \psi_k\right)_{H_{2m-k-1/2}} = \frac{1}{2\pi i}\int\limits_{\Gamma_\alpha} \left(g(\lambda)\, , R\left(\bar{\lambda}\right)\right) e^{\lambda t} dt \ .$$

Since the function $v(\lambda) = \left(g(\lambda)\, , R\left(\bar{\lambda}\right)\right)_H$ is an entire function and on Γ_α it holds the estimation $\|v(\lambda)\| \le c|\lambda|^{2m-1}$, then

$$v(\lambda) = \sum_{j=0}^{m-1} a_j \lambda^j \ .$$

Since

$$\int\limits_{\Gamma_2} v(\lambda)\, e^{\lambda t} d\lambda = 0 \ ,$$

then, for $t > 0$

$$\sum_{j=0}^{m-1} \left(u^{(k)}(t)\, , \psi_k\right)_{H_{2m-k-1/2}} = 0 \ .$$

Passing to the limit as $t \to 0$, we get that

$$\sum_{k=0}^{m-1} \|\psi_k\|_{H_{2m-k-1/2}} = 0 \ ,$$

i.e. all $\psi_k = 0$, $\ k = \overline{0, m-1}$. The theorem is proved.

References

[1]. Mirzoyev S.S. *The problems of the theory of solvability of boundary value problems for operator differential equations in a Hilbert space and related spectral problems.* Abstract of a doctor's thesis, 1994, Baku, 32 p. (Russian)

[2]. Humbataliyev R.Z. *On the completeness of elementary solutions of a class of operator-differential equations of a higher order.* Doklady NAN Azerb., 2002, v.LVIII, No5-6, pp.25-28. (Russian)

[3]. Keldysh M.V. *On the completeness of eigen functions of some classes of nonselfadjoint linear operators.* Uspekhi mat. nauk, 1971, v.XXVI, issue 4 (160), pp.15-41. (Russian)

Rovshan Z. Humbataliyev

Institute of Mathematics and Mechanics of NAS of Azerbaijan.

9, F.Agayev str., AZ1141, Baku, Azerbaijan.

Tel.: (99412) 439 47 20 (off.)

Received September 15, 2003; Revised January 27, 2004.

Translated by Mamedova Sh.N.