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ON SPECTRUM OF THREE-DIMENSIONAL BIHORMONIC OPERATOR WITH POINT INTERACTIONS

Abstract

In the present paper the spectrum of the operator $A=\Delta^2+\alpha_1\delta(x-x^{(1)})+$ $+\alpha_2\delta(x-x^{(2)})$ *in the space* $L_2(R^3)$ *is investigated. It is shown that the essential spectrum and absolutely continuous part of spectrum of the operator* A *coircide. The transcendental equation is obtained for the determination of negative eigen-values of the operator* A.

Consider in the space $L_2(R^3)$ the operator

$$A=\Delta^2+\alpha_1\delta(x-x^{(1)})+\alpha_2\delta(x-x^{(2)}) \tag{1}$$

with dense domain of determination

$$D(A)=\left\{u\in W_2^2(R^3)\,\middle|\,\Delta^2u+\alpha_1\delta(x-x^{(1)})u+\alpha_2\delta(x-x^{(2)})u\in L_2(R^3)\right\}. \tag{2}$$

Here Δ is a Laplacian three-dimensional operator, $W_2^2(R^3)$ is a Sobolev space, $\delta(x)$ is a delta function, $\alpha_1,\ \alpha_2\in R$ and $x^{(1)},\ x^{(2)}\in R^3$ are fixed points. Since $\alpha_1\delta(x-x^{(1)})+\alpha_2\delta(x-x^{(2)})\in W_2^{-\frac{3}{2}-\varepsilon}(R^3)$ $(\varepsilon>0)$ then A is a lower semibounded self-adjoint operator in the space $L_2(R^3)$ (see for ex.[1]).

In the present paper the explicit expression is found for the resolvent of the operator A. It is shown that the essential spectrum (limiting) and absolutely continuous part of a spectrum of the operator A coincide. The transandental equation is obtained for the determination of negative eigen-values of the operator A. The analogous results are obtained in [2] for the operator $\Delta^2+\alpha\delta(x)$ in $L_2(R^3)$. The two-dimensional biharmonic operator with δ potential is investigated in [3].

The following theorem is true.

Theorem 1. *The resolvent of the operator* A *is an integral operator in the space* $L_2(R^3)$

$$(R_z(A)f)(x)=\int\limits_{R^3}G(x,y;z)f(y)dy \quad (f\in L_2(R^3)).$$

The integral kernel $G(x,y;z)$ *at* $z=-\lambda^4$ $(\lambda>0), z\in\rho(A)$ *has the form*

$$G(x,y;-\lambda^4)=G_0\left(|x-y|\,;-\lambda^4\right)-\frac{4\sqrt{2}\pi\lambda}{d(\lambda)}\sum_{i,j=1}^{2}\left(1-\delta_{ij}\right)\alpha_i(\alpha_j+4\sqrt{2}\pi\lambda)\times$$

$$\times G_0\left(\left|x-x^{(i)}\right|;-\lambda^4\right)G_0\left(\left|x^{(i)}-y\right|;-\lambda^4\right)+$$

$$+\frac{\left(4\sqrt{2}\pi\lambda\right)^2\alpha_1\alpha_2}{d(\lambda)}G_0\left(\left|x^{(1)}-x^{(2)}\right|;-\lambda^4\right)\times$$

$$\times\sum_{i,j=1}^{2}(1-\delta_{ij})\,G_0\left(\left|x-x^{(i)}\right|;-\lambda^4\right)G_0\left(\left|x^{(j)}-y\right|;-\lambda^4\right). \tag{3}$$

Here δ_{ij} is a Kroneker symbol.

$$G_0\left(|x|;-\lambda^4\right)=\frac{1}{4\pi\lambda^2|x|}e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\frac{\lambda}{\sqrt{2}}|x|,$$

$$d(\lambda)=\left(\alpha_1+4\sqrt{2}\pi\lambda\right)\left(\alpha_2+4\sqrt{2}\pi\lambda\right)-32\pi^2\lambda^2\alpha_1\alpha_2G_0^2\left(\left|x^{(1)}-x^{(2)}\right|;-\lambda^4\right).$$

Proof. Let's find the resolvent of the self-adjoint operator A. We solve the equation

$$Au+\lambda^4u=f\quad (f\in L_2(R^3),\ \lambda>0) \tag{4}$$

in the space $L_2(R^3)$.

Since

$$\delta(x-x^{(k)})u=u(x^{(k)})\delta(x-x^{(k)}),\ k=1,2,$$

then we can write equation (4) in the following form

$$\Delta^2u(x)+\sum_{i=1}^{2}\alpha_iu(x^{(i)})\delta(x-x^{(i)})+\lambda^4u=f. \tag{5}$$

Applying Fourier transformation F to equation (5) and granting that

$$F[\Delta^2u]=|\xi|^4F[u],\ F[\delta(x-x^{(k)})]=e^{i(x^{(k)},\xi)},k=1,2,$$

we obtain

$$F[u]=\frac{1}{|\xi|^4+\lambda^4}F[f]-\sum_{k=1}^{2}\frac{\alpha_ku(x^{(k)})}{|\xi|^4+\lambda^4}e^{i(x^{(k)},\xi)}. \tag{6}$$

Now we apply the inverse Fourier transformation to equation (6) and use the known formulae

$$F^{-1}\left[\frac{1}{|\xi|^4+\lambda^4}\right]=\frac{1}{4\pi\lambda^2|x|}e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\frac{\lambda}{\sqrt{2}}|x|,$$

$$F^{-1}\left[\frac{e^{i(x^{(k)},\xi)}}{|\xi|^4+\lambda^4}\right]=\frac{1}{4\pi\lambda^2\left|x-x^{(k)}\right|}e^{-\frac{\lambda}{\sqrt{2}}\left|x-x^{(k)}\right|}\sin\frac{\lambda}{\sqrt{2}}\left|x-x^{(k)}\right|,\ (k=1,2),$$

$$F^{-1}\left[\frac{1}{|\xi|^4+\lambda^4}F[f]\right]=F^{-1}\left[\frac{1}{|\xi|^4+\lambda^4}\right]*f.$$

Then we obtain

$$u(x)=G_0\left(|x|;-\lambda^4\right)*f-\sum_{k=1}^{2}\alpha_ku(x^k)G_0\left(\left|x-x^{(k)}\right|;-\lambda^4\right). \tag{7}$$

We find $u(x^{(1)})$ and $u(x^{(2)})$. Write in (7) $x=x^{(1)}$ and $x=x^{(2)}$ consecutively.

Then using the relations

$$\lim_{x\to x^{(k)}} G_0\left(\left|x-x^{(k)}\right|;-\lambda^4\right)=\frac{1}{4\sqrt{2}\pi\lambda},\quad (k=1,2),$$

we obtain the linear system of equations with respect to $u(x^{(1)})$ and $u(x^{(2)})$

$$\left(1+\frac{1}{4\sqrt{2}\pi\lambda}\right)u(x^{(1)})+\alpha_2 G_0\left(\left|x^{(1)}-x^{(2)}\right|;-\lambda^4\right)u(x^{(2)})=$$

$$=\int\limits_{R^3} G_0\left(\left|x^{(1)}-y\right|;-\lambda^4\right)f(y)dy, \tag{8}$$

$$\alpha_1 G_0\left(\left|x^{(1)}-x^{(2)}\right|;-\lambda^4\right)u(x^{(1)})+\left(1+\frac{1}{4\sqrt{2}\pi\lambda}\right)u(x^{(2)})=$$

$$=\int\limits_{R^3} G_0\left(\left|x^{(2)}-y\right|;-\lambda^4\right)f(y)dy. \tag{9}$$

Solving the system of equations (8) and (9) we find $u(x^{(1)})$ and $u(x^{(2)})$.

Further, we put the found expressions $u(x^{(1)})$ and $u(x^{(2)})$ in (7). Then after simple transformations we obtain

$$u(x)=\int\limits_{R^3} G\left(x,y;-\lambda^4\right)f(y)dy,$$

where $G(x,y;-\lambda^4)$ has the representation (3). Hence and from (4) it follows that the resolvent $R_{-\lambda^4}(A)$ is an integral operator with the kernel $G(x,y;-\lambda^4)$. In the general case $z\in\rho(A)$ for obtaining the representations of the kernel $G(x,y;z)$ in (3) λ should be substituted by $\sqrt[4]{-z}$, moreover a regular branch of the root $\sqrt[4]{-z}$ is chosen. It equals to

$$\sqrt[4]{-z}=\sqrt[4]{r}e^{\frac{\varphi-\pi}{4}i},\ z=re^{i\varphi},\ 0<\varphi<2\pi.$$

Theorem 1 is proved.

The structure of a spectrum of the operator A is described by the following theorem.

Theorem 2. *The essential spectrum of the operator A coincide with an absolutely continuous part of the spectrum*

$$\sigma_{ess}(A)=\sigma_{ac}(A)=[0,+\infty). \tag{10}$$

If $\sigma_1>0,\ \alpha_2>0$, then the operator A has no eigen-values, if $\alpha_1\alpha_2<0$ then A has exactly one negative prime eigen-value $-\lambda_0^4$ where $\lambda_0>0$ is a unique positive root of the equation

$$\left(4\sqrt{2}\pi\lambda+\alpha_1\right)\left(4\sqrt{2}\pi\lambda+\alpha_2\right)-\frac{2\alpha_1\alpha_2}{\lambda^2\left|x^{(1)}-x^{(2)}\right|^2}e^{-\sqrt{2}\lambda\left|x^{(1)}-x^{(2)}\right|}\times$$

$$\times \sin^2 \frac{\lambda}{\sqrt{2}} \left| x^{(1)} - x^{(2)} \right| = 0. \tag{11}$$

In the case $\alpha_1 < 0, \ \alpha_2 < 0$ *the operator* A *has exactly two negative prime eigen-values,* $-\lambda_1^4$ *and* $-\lambda_2^4$*, where* $\lambda_1 > 0, \ \lambda_2 > 0$ *are the positive roots of equation (11).*

Proof. Let A_0 be a minimal operator in the $L_2(R^3)$ generated by the expression $\Delta^2 u$. It is evident that A_0 is a self-adjoint non-negative operator and resolvent $R_z(A_0)$ is an integral operator in $(z \notin [0, +\infty))$ in $L_2(R^3)$ with the kernel $G_0(x, y; z)$. The relations

$$\sigma(A_0) = \sigma_{ess}(A_0) = \sigma_{ac}(A_0) = [0, +\infty)$$

are true.

Denote

$$B = (A + \lambda_0^4 E)^{-1} - (A_0 + \lambda_0^4 E)^{-1}, \ (\lambda_0 > 0, \ -\lambda_0^4 \in \rho(A) \cap \rho(A_0)).$$

It is obvident that B is an integral operator in $L_2(R^3)$ with the integral kernel $k(x, y) \in L_2(R^3 \times R^3)$ where

$$k(x, y) = G(x, y; -\lambda_0^4) - G_0(|x - y|; -\lambda_0^4).$$

Therefore B is a Hilbert-Schmidt operator and therefore it is compact. By the weil theorem ([4], theorem XIII.14) the essential spectrums of the operators A and A_0 coincide, moreover,

$$\sigma_{ess}(A) = \sigma_{ess}(A_0) = [0, +\infty). \tag{12}$$

Further from the representation (3) it follows that the difference of the resolvents, $R_{-\lambda_0^4}(A) - R_{-\lambda_0^4}(A_0)$ is a finite dimensional operator. According to the known theorem ([5], ch. X, theorem 4.2) the absolutely continuous parts of the spectrums A and A_0 coincide. Hence and from (12) we obtain (10).

We find the negative eigen-values of the operator A. Let $-\lambda^4 (\lambda > 0)$ be an eigen-value of the operator A. Then

$$Au + \lambda^4 u = 0.$$

Assume $f = 0$ in (7)

$$u(x) = -\sum_{k=1}^{2} \alpha_k u(x^{(k)}) G_0 \left(\left| x - x^{(k)} \right|; -\lambda^4 \right). \tag{13}$$

Further, assuming $f = 0$ in (8) and (9) we obtain the system of the equations

$$\begin{cases} \left(1 + \dfrac{\alpha_1}{4\sqrt{2}\pi\lambda}\right) u(x^{(1)}) + \alpha_2 G_0 u(x^{(2)}) = 0, \\ \alpha_1 G_0 u(x^{(1)}) + \left(1 + \dfrac{\alpha_2}{4\sqrt{2}\pi\lambda}\right) u(x^{(2)}) = 0. \end{cases} \tag{14}$$

From the representation (13) it follows that for $u(x) \not\equiv 0$ it is necessary and sufficient that the system of equations (14) have non-zero solution relative to $u(x^{(1)})$ and $u(x^{(2)})$. So, the determinant of system (14) have to be equal to zero, i.e.,

$$\left(1+\frac{\alpha_1}{4\sqrt{2}\pi\lambda}\right)\left(1+\frac{\alpha_2}{4\sqrt{2}\pi\lambda}\right)-\alpha_1\alpha_2 G_0^2=0.$$

Allowing for the representation for G_0 we can write this equation in the form (11).

Let's now investigate equation (11). Denote

$$f(\lambda)=\left(4\sqrt{2}\pi\lambda+\alpha_1\right)\left(4\sqrt{2}\pi\lambda+\alpha_2\right),$$

$$g(\lambda)=\frac{2\alpha_1\alpha_2}{\lambda^2\left|x^{(1)}-x^{(2)}\right|^2}e^{-\sqrt{2}\lambda\left|x^{(1)}-x^{(2)}\right|}\sin^2\frac{\lambda}{\sqrt{2}}\left|x^{(1)}-x^{(2)}\right|.$$

The following cases are possible: 1) $\alpha_1>0,\ \alpha_2>0$; 2) $\alpha_1\alpha_2<0$; 3) $\alpha_1<0$, $\alpha_2<0$.

Consider the case 1). It is evident that at $\lambda>0$ and $\alpha_1,\ \alpha_2>0$, the inequality

$$f(\lambda)>\alpha_1\alpha_2 \tag{15}$$

is true.

Further, by the inequalities

$$|\sin x|<x,\quad e^{-x}<1,\ (x>0),$$

we have

$$g(\lambda)<\alpha_1\alpha_2. \tag{16}$$

From inequalities (15) and (16) it follows that equation (11) has no positive solutions by λ.

In cases 2) and 3) it is convenient to determine the amount of the roots of equation (11) by the graphic method, i.e., as an abscissa of intersection points of graphs of the functions $f(\lambda)$ and $g(\lambda)$.

Consider the case $\alpha_1 a_2<0$. It is easy to check that the points

$$\lambda_k=\frac{\sqrt{2}\pi k}{\left|x^{(1)}-x^{(2)}\right|},\ k=1,2,...,x^{(1)}\neq x^{(2)},$$

are the maximum points, and

$$\lambda'_k=\frac{\sqrt{2}\mu_k}{\left|x^{(1)}-x^{(2)}\right|},\ k=1,2,...,$$

where $\mu_k>0$ is a solution of the equation $tg\mu=\dfrac{\mu}{\mu+1}$, are minimum points of the functions $g(\lambda)$, moreover

$$g(\lambda_k)=0,\ g(\lambda'_k)=\frac{\alpha_1\alpha_2}{\mu_k^2}e^{-2\mu_k}\sin^2\mu_k,\ k=1,2,....$$

The other characteristic properties of the function $g(\lambda)$ necessary for the construction of a graph of this function are investigated in an ordinary way. Further, the graph of the function $f(\lambda)$ is a parabola and $\lambda_0 = -\dfrac{\alpha_1+\alpha_2}{8\sqrt{2}\pi}$ is a minimum point moreover

$$f(\lambda_0) = -\frac{(\alpha_2-\alpha_1)^2}{4}.$$

Using these data it is easy to show that the graph of the functions $f(\lambda)$ and $g(\lambda)$ at $\lambda > 0$ intersect only at one point.

Acting similarly we obtain that in the case $\alpha_1 < 0$, $\alpha_2 < 0$, the graphs of the functions $f(\lambda)$ and $g(\lambda)$ for $\lambda > 0$ intersect at two points. Consequently, in this case equation (11) has two positive roots. Theorem 2 is proved.

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