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ON APPLICATION OF FOURIER-WIENER TRANSFORM TO THE LOCAL LIMIT THEOREM IN A HILBERT SPACE

Abstract

Let on the Hilbert space X be given two measures μ_0 and μ, moreover μ_0 is a Goursat measure, and μ is absolutely continuous relative to μ_0 with the density from $L_2(X,\mu_0)$. Let $f^{(n)}(x)$ be a density of n-fold convolution μ^{*n} relatively to μ_0. The conditions are given when $f^{(n)}(x)$ is converges in $L_2(X,\mu_0)$ as $n\to\infty$.

1. Introduction. In this paper densities of convoluted distributions absolutely continuous with respect to some Gaussian measure in a Hilbert space are studied. We can regard the obtained result on convergence in quadratic sequence of densities as analogue of classical local limit theorem. The fact is that there are no measures having all the properties of Lebesque measure in Hilbert space, therefore the used apparatus of inversion formula essentially differs from classical one, and as sequence, local theorems in classical formulation can not be transferred to infinite-dimensional case. In this paper one representation of densities by characteristic functional constructed by Skorokhod [1], theory of Fourier-Wiener transformation developed by Kameron and Martin [4], [5], [6] in the space of continuous functions [2] and transferred to Hilbert space in [2] are used.

2. Notation, definitions and the known facts. Let X be a separable Hilbert space with scalar product $(x,y)\,,\ x,y\in X$. Let two measures μ_0 and μ be given on σ-algebra of Borel sets, where μ_0 is Gaussian measure with characteristic functional, $\varphi_0(z)=\exp\left\{-\frac{1}{2}(Bz,z)\right\}$ and μ is absolutely continuous with respect to μ_0 with the density $f(x)$ and characteristic functional.

$$\varphi(z)=\int e^{i(z,x)}\mu dx=\int e^{i(z,x)}f(x)\,\mu_0(dx)\,.$$

Let X_- and X_{--} be completions of X in metric, generated by scalar product $(x,y)_-=(Bx,y)$ and $(x,y)_{--}=(Sx,y)_-$ respectively, where S is some positive symmetric operator in X_- with finite trace. In [1] it is shown that on X_{--} there exists the Gaussian measure μ_0^* with the characteristic functional $\varphi_0^*(z)=\exp\left\{-\frac{1}{2}\left(B^{-1}z,z\right)\right\},\quad z\in B^{1/2}X$ and

$$f(x)=\lim_{n\to\infty}\exp\left\{\frac{1}{2}\left(B^{-1}P_nx,P_nx\right)\right\}\int\limits_{X_{--}}e^{-i(P_nx,P_nz)}\frac{\varphi(P_nz)}{\varphi_0(P_nz)}\mu_0^*(dz)\ ,\qquad(1)$$

where the convergence is almost everywhere with respect to measure μ_0, and P_n is an operator of orthogonal designing on infinite-dimensional subspace X_n stretched to first n eigen vectors of the operator B. From constructions in [1] it follows that expression after sign lim in the right hand side of (1) is $f(P_nx)$.

From [3], [4], [5] we cite the facts which in the case of the Hilbert space $L_2(X,\mu_0)$ are look like as: the class of "entire analytic" functions is chosen in $L_2(X,\mu_0)$. These are such functions $f(x)\in L_2(X,\mu_0)$, where $f(x+\lambda y)$ is entire analytic with respect to complex variable λ for any fixed $x,y\in X$ extendable on complexification of X. For the functions from this class the Fourier-Wiener transformation

$$\hat{f}(z)=\int\limits_X f\left(\sqrt{2}x+iz\right)\mu_0(dx)$$

is determined and the Plancherel equality

$$\int |f(x)|^2\mu_0(dx)=\int\left|\hat{f}(z)\right|^2\mu_0(dz) \tag{2}$$

holds.

This transformation is extended on all $L_2(X,\mu_0)$ with equality (2). We also will denote by $\hat{f}(z)$ the Fourier-Wiener transformation for any function $f(x)\in L_2(X,\mu_0)$.

The following expansions hold

$$f(x)=\underset{N\to\infty}{l.i.m} f_N(x),\quad \hat{f}(z)=\underset{N\to\infty}{l.i.m}\hat{f}_N(z),$$

where

$$f_N(x)=\sum_0^N A_{\bar{m}_N}\Psi_{\bar{m}_N}(x)\ ,\quad \hat{f}_N(z)=\sum_0 i^{|\bar{m}_N|}A_{\bar{m}_N}\Psi_{\bar{m}_N}(z)\ ,$$

$$A_{\bar{m}_N}=\int f(x)\Psi_{\bar{m}_N}(x)\mu_0(dx),\quad \bar{m}_N=(m_1m_2...m_N)$$

$m_i=0,1,2,...,N,\quad |\bar{m}_N|=\sum\limits_{k=1}^N m_k$ and the summation is conducted on all coordinates of the vector $\bar{m}_N$. Here $\Psi_{\bar{m}_N}(x)$ is Fourier-Hermite polynomial and is determined as follows

$$\Psi_{\bar{m}_p}(x)=\Psi_{m_1m_2...m_p}(x)=\Phi^1_{m_1}(x)\Phi^2_{m_2}(x)...\Phi^p_{m_p}(x)\ ,$$

$$\Phi^k_{m_k}(x)=\frac{H_{m_k}\left[(x,e_k)/\lambda_k\right]}{\sqrt{m_k!\lambda_k}}\ ,\qquad H_m(u)=(-1)^m e^{-\frac{u^2}{2\lambda}}\frac{d^m}{du^m}e^{-\frac{u^2}{2\lambda}}$$

where $\{e_i\}$ and $\{\lambda_i\}$ are systems of eigen elements and eigen numbers of the operator B respectively. The equality $\hat{\Psi}_{\bar{m}_N}(z)=i^{|\bar{m}_N|}\Psi_{\bar{m}_N}(z)$ holds.

We now introduce some operations and quantities whose studying is the goal of this paper.

Denote by μ_0^{*n} and μ^{*n} n-fold convolutions of measure μ_0 and μ, respectively. It is clear that μ^{*n} is absolutely continuos with respect to μ_0^{*n}.

The characteristic functions of the measures μ_0^{*n} and μ^{*n} are equal to $[\varphi_0(z)]^n$ and $[\varphi(z)]^n$, respectively. The characteristic function of the measure $\mu_0^{*n}(\cdot/\sqrt{n})$ is equal to

$$[\varphi_0(z/\sqrt{n})]^n=\left[\exp\left\{-\frac{1}{2}\left(Bz/\sqrt{n},z/\sqrt{n}\right)\right\}\right]^n=\exp\left\{-\frac{1}{2}(Bz,z)\right\}\ .$$

It means that $\mu_0^{*n}(\cdot/\sqrt{n}) = \mu_0(\cdot)$ for any $n \geq 1$. Hence, $\mu^{*n}(\cdot/\sqrt{n})$ is absolutely continuous with respect to μ_0 for any $n \geq 1$. Denote by $f^{(n)}(x)$ density $\mu^{*n}(\cdot/\sqrt{n})$ with respect to μ_0. Then by definition

$$\left[\varphi\left(z/\sqrt{n}\right)\right]^n = \int e^{i(z,x)} f^{(n)}(x)\,\mu_0(dx)\ . \tag{3}$$

In [3] it is shown that

$$\frac{\varphi(z)}{\varphi_0(z)} = \hat{f}(Bz)\,, \quad z \in X_{-\,-} \tag{4}$$

Hence, it follows hat

$$\frac{\left[\varphi\left(z/\sqrt{n}\right)\right]^n}{\varphi_0(z)} = \left[\frac{\varphi\left(z/\sqrt{n}\right)}{\varphi_0\left(z/\sqrt{n}\right)}\right]^n = \left[\hat{f}\left(\frac{Bz}{\sqrt{n}}\right)\right]^n, \quad z \in X_{-\,-} \tag{5}$$

On the other hand from (3) and (4) it follows that

$$\frac{\left[\varphi\left(z/\sqrt{n}\right)\right]^n}{\varphi_0(z)} = \left[\hat{f}^n(Bz)\right]^n, \quad z \in X_{-\,-} \tag{6}$$

Then form (5) and (6) it follows

$$\left[\hat{f}^n(Bz)\right]^n = \left[\hat{f}\left(\frac{Bz}{\sqrt{n}}\right)\right]^n, \quad z \in X_{-\,-}$$

Inversion formula (1) for density $f^{(n)}(x)$ gets the form

$$f^{(n)}(x) = \lim_{N\to\infty} \exp e^{\frac{1}{2}\left(B^{-1}P_n x, P_n x\right)} \int\limits_{X_{-\,-}} e^{-i(P_n x, P_n z)} \left[\hat{f}\left(\frac{BP_n z}{\sqrt{n}}\right)\right]^n \mu_0^*(dz)\ . \tag{7}$$

and for any $n \geq 1$ and $N \geq 1$ the Plancherel equalities

$$\int \left|f^{(n)}(x)\right|^2 \mu_0(dx) = \int \left|\left[\hat{f}\left(z/\sqrt{n}\right)\right]^n\right|^2 \mu_0(dz) \tag{8}$$

$$\int \left|f^{(n)}(P_N x)\right|^2 \mu_0(dx) = \int \left|\left[\hat{f}\left(P_N z/\sqrt{n}\right)\right]^n\right|^2 \mu_0(dz) \tag{8'}$$

hold. In [3] it is shown that for $g(x) \in L_2(X, \mu_0)$ the equality

$$\int\limits_X |g(x)|^2 \mu_0(dx) = \int\limits_{X_{-\,-}} |\hat{g}(Bz)|^2 \mu_0^*(dz) \tag{8''}$$

holds.

3. Theorems on convergence of densities.

Theorem 1. *Let the following conditions be satisfied*

1. $f(x) \in L_2(X, \mu_0)$ and $\hat{f}(z) = \exp\{V(z)\}$.

2. $|V(\alpha z)| \leq C\alpha^{2+\delta}\theta(\alpha, z)\,,\ C > 0$ *at some* $\delta > 0$, *where* $\theta(\alpha, z)$ *is continuous with respect to* α *at zero for each* z *and* $\theta(\alpha, P_n z) \leq \theta(\alpha, z)\,,\ N \geq 1$.

3. $\operatorname{Re} V(z) \leq \frac{1}{2}\beta\|z\|^2$ *moreover* $\beta\|B\|^2 < \gamma$, *where* $\gamma > 0$ *such that*

$$\int \exp\left(\gamma\|z\|^2\right) \mu_0(dz) < \infty.$$

Then

$$f^{(n)}(x) \to 1 \quad in \quad L_2(X, \mu_0), \quad as \quad n \to \infty.$$

Proof. From conditions 1 and 2 it follows that

$$\left| \left[\hat{f}\left(P_N Bz/\sqrt{n}\right) \right]^n - 1 \right| \le \exp\left\{ n \left| V\left(P_N Bz/\sqrt{n}\right) \right| \right\} - 1 \le$$

$$\le \exp\left\{ n^{-\delta/2} \theta\left(\frac{1}{\sqrt{n}}, P_N Bz \right) \right\} - 1 \exp\left\{ n^{-\delta/2} \theta\left(\frac{1}{\sqrt{n}}, Bz \right) \right\} - 1 \, .$$

Hence, it follows that

$$\left| \left[\hat{f}\left(P_N Bz/\sqrt{n}\right) \right]^n - 1 \right| \to 0, \quad n \to \infty \tag{9}$$

uniformly with respect to N at each point $z \in X_{-\,-}$. From condition 3 we have

$$\left| \left[\hat{f}\left(P_N Bz/\sqrt{n}\right) \right]^n - 1 \right| \le e^{\operatorname{Re} V(P_n Bz)} + 1 \le e^{\frac{1}{2}\beta \|Bz\|^2} + 1 \le e^{\frac{1}{2}\gamma \|z\|^2} + 1 \, .$$

Then from $(8'')$ it follows that $\exp\left\{ \frac{1}{2}\beta \|Bz\|^2 \right\} \in L_2(X_{-\,-} \mu_0^*)$. Consequently, the convergence in (9) is dominated. Allowing for $(8')$ and $(8'')$ for given $\varepsilon > 0$ there will be found n_0 such that for all $n > n_0$ and $N \ge 1$ we will have

$$\int \left| f^{(n)}(P_N x) - 1 \right|^2 \mu_0 dx = \int\limits_{X_{-\,-}} \left| \left[\hat{f}\left(P_N Bz/\sqrt{n}\right) \right]^n - 1 \right|^2 \mu_0^*(dz) < \varepsilon \, .$$

Passing to the limit in the right hand side as $N \to \infty$ we obtain

$$\int \left| f^{(n)}(x) - 1 \right|^2 \mu_0 dx < \varepsilon$$

for all $n > n_0$, which proves the theorem.

Theorem 2. *Let the following conditions be satisfied*

1) $f(x) \in L_2(X, \mu_0)$, $\left| \hat{f}(z) \right| \le \exp\left\{ \frac{\alpha}{2} \|z\|^2 \right\}$, *where* α *is such that* $\int \exp\left\{ \alpha \|z\|^2 \right\} \mu_0(dx) < \infty$

2) $A_{\bar{m}_N} = 0$ *for all the vectors* $\bar{m}_N = (m_1, m_2, ..., m_N)$ *satisfying the following conditions.*

a) if odd components is at most one.

b) If any two components are odd and at least one is more than unit, th others are even and at least one is more that zero for any $N > 1$. *Then* $f^{(n)}(x)$ *converges in* $L_2(X, \mu_0)$ *as* $n \to \infty$.

Proof. Denote by

$$\hat{f}_N(z) = \sum_{|m_N|=1}^{N}{}' i^{|m_N|} A_{\bar{m}_N} \Psi_{\bar{m}_N}(z) \ , \qquad \hat{f}''_N(z) = \sum_{|m_N|=1}^{N}{}'' i^{|m_N|} A_{\bar{m}_N} \Psi_{\bar{m}_N}(z)$$

where the prime means summation with respect to vector of the form (0...010...010...0), and two primes means summation with more that two odd components. These sums represent partial sums of subseries of Fourier-Hermite series of the functions $\hat{f}(z)$, therefore converge in $L_2(X, \mu_0)$ to some functions $\hat{f}(z)$ and $\hat{f}''(z)$

from $L_2(X,\mu)$. Since $\hat{f}'_N(z/\sqrt{n}) = \hat{f}'_N(z)/n$ and $\hat{f}''_N(z/\sqrt{n}) = \hat{f}''_N\left(z,\frac{1}{\sqrt{n}}\right)/(\sqrt{n})^3$, then $\hat{f}'(z)$ and $\hat{f}''(z)$ also have these properties, i.e. $\hat{f}'_N(z/\sqrt{n}) = \hat{f}'_N(z)/n$, $\hat{f}''_N(z/\sqrt{n}) = \hat{f}''_N\left(z,\frac{1}{\sqrt{n}}\right)/(\sqrt{n})^3$. Since $\hat{f}_N(z) = 1 + \hat{f}'_N(z) + \hat{f}''_N(z)$ and $\left[\hat{f}_N\left(\frac{z}{\sqrt{n}}\right)\right]^n = \left[1 + \frac{\hat{f}'_N(z)}{n} + \frac{\hat{f}''_N\left(z,\frac{1}{\sqrt{n}}\right)}{n\sqrt{n}}\right]^n$, then as $N\to\infty$ we obtain

$$\hat{f}(z) = 1 + \hat{f}'(z) + \hat{f}''(z)\,, \quad \left[\hat{f}_N\left(\frac{z}{\sqrt{n}}\right)\right]^n = \left[1 + \frac{\hat{f}'(z)}{n} + \frac{\hat{f}''\left(z,\frac{1}{\sqrt{n}}\right)}{n\sqrt{n}}\right]^n\,,$$

where the equality are satisfied almost everywhere with respect to measure μ_0 and $\hat{f}''\left(z,\frac{1}{\sqrt{n}}\right)$ is almost everywhere finite function at any $n\geq 1$. Hence, it follows that $\left[\hat{f}(z,\sqrt{n})\right] \to \exp\left\{\hat{f}'(z)\right\}$, as $n\to\infty$ almost everywhere with respect to measure μ_0. From condition 1 of the theorem it follows that

$$\left|\left[\hat{f}\left(\frac{z}{\sqrt{n}}\right)\right]^n\right| \leq \left[e^{\frac{\alpha}{2}\|z/\sqrt{n}\|^2}\right]^n = e^{\frac{\alpha}{2}\|z\|^2}\,, \quad z\in X$$

Consequently, $\exp\left\{\hat{f}'(z)\right\} \in L_2(X,\mu_0)$ ..

It is known [5] that the Fourier-Wiener transformation in $L_2(X,\mu)$ has inversion. Therefore, denote by $g(x)$ such fucntion from $L_2(X,\mu_0)$ which by its Fourier-Wiener transformation has $\exp\left\{\hat{f}'(z)\right\}$. Then the Plancherel equality

$$\int\left|f^{(n)}(x) - g(x)\right|^2 \mu_0 dx = \int\left|\left[\hat{f}\left(\frac{z}{\sqrt{n}}\right)\right]^n - e^{\hat{f}'(z)}\right|^2 \mu_0 dz$$

holds.

In the right hand side we have dominated convergence to zero of integrand, consequently, the integral in the right hand side of equality converges to zero. Hence $f^{(n)}(x) \to g(x)$ in mean square. When $\exp\left\{\hat{f}'(z)\right\} = 1$ then $g(x) = 1$ and $f^{(n)}(x) \to 1$ (m.s.) as $n\to\infty$.

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