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ON THE BEHAVIOR OF SOLUTIONS OF MIXED BOUNDARY-VALUE PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS

Abstract

In this paper we investigated the local behavior of solutions of mixed boundary problems for quasilinear elliptic equations of second order with general structure

I. Introduction

In this paper we investigate the problem

$$\frac{d}{dx_i}a_i\left(x,u,u_x\right)+a\left(x,u,u_x\right)=T, \tag{1}$$

$$u\left|_{\Gamma_1}=0,\ a_i\left(x,u,u_x\right)\cos\left(n,x_i\right)\right|_{\Gamma_2}=0, \tag{2}$$

in the domain Ω. Let Ω be an open set in R^n with the boundary $\partial\,\Omega=\Gamma_1\cup\Gamma_2$ and let $1<m\leq n$ be a fixed number, and $0\in\bar{\Gamma}_1\cap\Gamma_2$. Moreover we suppose that domain Ω satisfyies isoperimetric conditions. Here T is a distribution that will be specified below. The case T is a measure arises in the study of variational inequalities. Hence consequently that u is weak solution of inequality

$$\int\limits_{\Omega}\left[a_i\left(x,u,u_x\right)\varphi_{x_i}+a\left(x,u,u_x\right)\varphi\right]dx\geq 0 \tag{3}$$

for all nonnegative $\varphi\in W^1_{m,0}\left(\Omega\right)$. Here $W^1_{m,0}(\Omega)$ is a closure in $W^1_m(\Omega)$ of functions from $C_0^\infty(\partial\,\Omega\backslash\Gamma_2)$. Consequently u is a solution of the equation

$$-\frac{d}{dx_i}a_i\left(x,u,u_x\right)+a\left(x,u,u_x\right)=\mu, \tag{4}$$

where μ is some nonnegative measure.

We assume the functions a_i and a Borel functions defined on $\Omega\times(m,M)\times R^n$, $-\infty<m\leq M<+\infty$, that satisfy the following structure: for a.e. $x\in\Omega,\ u\in(m,M)\,,\ \forall p\in R^n$,

$$a_i\left(x,u,p\right)p_i\geq\mu_0(x)\left|p\right|^m-d_0(x)\left|u\right|^m-\nu_1(x), \tag{5}$$

where $1<m<n,\ \mu_0$ is a positive continuous function on Ω, and $\nu_1\in L^{n/m+\varepsilon_1}_{loc}(\Omega)$, with $\varepsilon_1>0,\ d_0(x)\in L^\infty_{loc}(\Omega),\ d_0\geq 0$,

$$\left|a_i\left(x,u,p\right)\right|\leq c_0(x)\left|p\right|^{m-1}-b_0(x)\left|u\right|^{m-1}+g_0(x), \tag{6}$$

for $c_0\in L^\infty_{loc}(\Omega),\ c_0\geq 0$, and $g_0\in L^{n/(m-1)+\varepsilon_2}\left(\Omega\right),\ \varepsilon_2>0,\ b_0(x)\in L^\infty_{loc}(\Omega)$, $b_0(x)\geq 0$,

$$\left|a(x,u,p)\right|\leq c_1(x)\left|p\right|^m+d_0(x)\left|u\right|^{m-1}+k_0(x), \tag{7}$$

where $c_1 \in L^{\infty}_{loc}(\Omega),\ c_1 \geq 0,\ k_0(x) \in L^{n/m+\varepsilon_3,}_{loc}(\Omega),\ \varepsilon_3 > 0$. In the sequel, we will also consider the structure condition

$$|a(x,u,p)| \leq c_1(x)\,|p|^{m-1} + d_0(x)\,|u|^{m-1} + k_0(x). \tag{8}$$

2. Preliminaries. For $m \geq 1$ we denote by m' the conjugate of m. Choose numbers q and q_1 such that

$$\frac{n}{m-1} < q < q_1\left(\frac{2nm'}{mq_1+nm'}\right) < q_1. \tag{9}$$

We also consider a distribution T in (1) such that $T \in W^{-1}_{q_1}\left(\Omega^1\right)$, where Ω^1 fix a bounded open set, relatively compact in Ω.

We denote $\Omega_r = \Omega \cap B_r(x_0),\ B_r = \{x : |x-x_0| < r\}$. By the assumptions on $\nu_1,\ g_0,\ k_0$

$$T, k_0 \in W^{-1}_{q_1}(\Omega^1),\ g_0 \in L_{q_1}(\Omega^1),\ \nu_1 \in L_{q_1/m'}(\Omega'). \tag{10}$$

Let $\varepsilon_0 = \pm 1$. Since $\varepsilon_0 T + k_0 \in W^{-1}_{q_1}(\Omega^1)$, there exists $k \in L^q(\Omega^1, R^n)$ such that $\varepsilon_0 T + k_0 = div k$. We denote $\mu_0 = \min\limits_{\Omega^1}\mu_0(x), c_0 = \underset{\Omega^1}{ess}\sup c_0(x), c_1 = \underset{\Omega^1}{ess}\sup c_1(x)+1$. We consider $x_0 \in \Omega^1$ and for all $r > 0$ such that Ω_{4r} is contained in $\overline{\Omega^1}$. We define $\rho(r) = r^{1-n/(m-1)q}I(\Omega^1)$ and $I(\Omega^1) = \|T\|_{W^{-1}_q(\Omega^1)} + \|k_0\|_{L^{n/m}(\Omega^1)} + \|g_0\|_{L^{n/m-1}(\Omega^1)} + \|\nu_1\|_{L^{n/(m-1)}(\Omega^1)}$. Define also the function

$$b_r(x) = \frac{1}{\rho^m(r)}\left[(k+g_0(x))^{m'} + \nu_1(x)\right] \text{ if } I(\Omega^1) \neq 0$$

and $b_r(x) = 0$ if $I(\Omega^1) = 0$. By an application of Hōlders inequality, observe that there exists a constant c_2 such that for all $x_0 \in \Omega$, and all $r > 0$ with $\Omega_r \subset \overline{\Omega^1}$ we have

$$\|b_r\|_{L^{q/m'}(\Omega_r)} \leq c_2. \tag{11}$$

We obtain weak Harnack inequalities for (1), (2) with T as in (10) analogous to those obtained in [1] where $T = 0$.

Let $\mu_1 > 0,\ \mu_2 > 0$ be arbitrary. For $\beta \in R$ and $t > 0$ define

$$\sigma(t) = \frac{1}{\mu_1}\exp\left(\frac{\mu_2}{\mu_1}(sign\beta)\,t\right)t^{\beta}. \tag{12}$$

For $x_0 \in \Omega^1$ let $\eta \in C_0^{\infty}(\Omega)$ be such that $0 \leq \eta \leq 1$ and $spt\ \eta \subset \Omega_r \subset \overline{\Omega^1}$. Let $0 < \lambda < 1$, where λ tends to zero at the end of the proof. Define $\bar{u} = u + \rho(r) + \lambda$, $\gamma = \beta + m - 1,\ \beta \neq 0$,

$$W(x) = \begin{cases} \overline{u}^{\gamma/m}, & \gamma \neq 0, \\ \log \overline{u}, & \gamma = 0. \end{cases}$$

Lemma 1. *There exists a constant $C > 0$ such that*

$$\int\limits_{\Omega_r} |\eta \nabla u|^m\, dx \leq C\,|\gamma|^m\, h_0(|\beta|)\int\limits_{\Omega_r}(b_r\eta^m + |\nabla\eta|^m)\,W^m(x)dx,$$

under $\gamma \neq 0, \int\limits_{\Omega_r} |\eta\nabla u|^m\, dx \le Ch_0(|\beta|) \int\limits_{\Omega_r} (b_r\eta^m + |\nabla\eta|^m)\, dx$ *under* $\gamma = 0$, *where* $h_0(|\beta|) = 1 + 1/|\beta| + 1/|\beta|^m + 1\,|\beta|^{m'}$.

Proof. Define $\varphi = \eta^m u\ sign\ \beta\ \sigma(\bar u)$, where σ is the function defined in relation (12) with $\mu_1 = \underset{\overline{\Omega^1}}{\min}\mu_0(x),\ \mu_2 = \underset{\Omega^1}{ess\sup}\, c_1(x) + 1$.

Since u is locally bounded, we can find $c > 0$ such that

$$0 < \sigma(\bar u) \le c\,|\bar u|^{\beta},\ |\sigma'(\bar u)| \le c\,|\beta|\,\bar u^{\beta-1} + c\bar u^{\beta},$$

$$(sign\beta)\,\mu_0\sigma'(\bar u) - c\sigma(\bar u) \ge c\,|\beta|\,\bar u^{\beta-1}. \tag{13}$$

With this choice of φ and account locally bounded u employed in (1), (2) with its structure (5)-(8), we obtain

$$\int\limits_{\Omega_r} |\nabla u|^m \left[\mu_1\sigma'(\bar u) sign\beta - \mu_2\sigma(\bar u)\right]\eta^m dx \le C_1 \int\limits_{\Omega_r} |\nabla u|\,\eta^{m-1}\,|\nabla u|^{m-1}\sigma(\bar u)dx +$$

$$+C_2\int\limits_{\Omega_r} \sigma(\bar u)\eta^{m-1}g_0(x)\,|\nabla\eta|\,dx + C_3\int\limits_{\Omega_r} |\sigma'(\bar u)|\,\nu_1\eta^m dx + C_4\int\limits_{\Omega_r}\sigma(\bar u)\eta^m k_0(x)dx +$$

$$+C_5\int\limits_{\Omega_r}\sigma(\bar u)\gamma^{m-1}C_0(x)\,|\nabla\eta|\,dx + \langle T,\varphi\rangle. \tag{14}$$

In (14) two terms can be written as

$$\int\limits_{\Omega_r}\sigma(\bar u)\eta^m k_0(x)dx + \langle T,\varphi\rangle = \langle K_0 + (sign\beta)T, \eta^m\sigma(\bar u)\rangle.$$

Using relation (10) we can find $k \in L^q(\Omega^1)$ such that if $k_0 + (sign\beta)T = divk$, where k does not depend on β. Using (13) we obtain

$$|\beta|\int\limits_{\Omega_r}\bar u^{\beta-1}\,|\nabla u|^m\,\eta^m dx \le C_1\int\limits_{\Omega_r}|\nabla u|\,\eta^{m-1}\,|\nabla u|^{m-1}\,\bar u^{\beta}dx +$$

$$+C_2\int\limits_{\Omega_r}\bar u^{\beta}\eta^{m-1}g_0(x)\,|\nabla u|\,dx + C_3\,(1+|\beta|)\int\limits_{\Omega_r} \le \bar u^{\beta-1}\nu_1\eta^m dx +$$

$$+C_4\int\limits_{\Omega_r}\bar u^{\beta}\eta^m k_0(x)dx + C_5\,|\beta|\int\limits_{\Omega_r}\bar u^{\beta-1}\eta^{m-1}kdx + C_6\int\limits_{\Omega_r}|\nabla u|\,\eta^m\bar u^{p-1}kdx. \tag{15}$$

The right side in (15) can be estimated in a manner to [2], we obtain the conclusion of the lemma.

Theorem 1. *Let* $u \in W^1_{m,0,loc}(\Omega)$ *be a nonnegative, locally bounded subsolution of (1), (2) with structure (5)-(8) and* $T \in W^{-1}_{q_0,loc}(\Omega)$, $q_0 > n/(m-1)$. *Then for* $r > 0$ *such that* $\Omega_r \subset \Omega^1$ *and for all* $m_0 > m-1$ *we have*

$$\sup_{\Omega_r} u \le C\left(r^{-n/m_0}\,\|u\|_{L_{m_0}(\Omega_{2r})} + \rho(r)\right), \tag{16}$$

constant C depends only on the structure, the bound on u, and the fixed set Ω^1.

Theorem 2. *Let $u\in W^1_{m,0,loc}(\Omega)$ be a nonnegative, locally bounded supersolution of (1), (2) with structure (5)-(8). Then for $r>0$ such that $\Omega_r\subset\overline{\Omega^1}$, and all m_0 such that $0<m_0<\hat{n}\,(m-1)\,/\,(\hat{n}-m)$ we have*

$$r^{-n/m_0}\,\|u\|_{L_{m_0}(\Omega_{2r})}\leq C\left(\min_{\Omega_r}u+\rho(r)\right),\tag{17}$$

here $\hat{n}=n$ if $m<n$, $n<\hat{n}<q\,(m-1)$ if $m=n$.

The proofs theorem 1 and theorem 2 it is possible to obtain with the basic energy inequality as in [2].

Theorem 1 and theorem 2 field the Harnack inequality

$$\sup_{\Omega_r}u\leq C\left(\inf_{\Omega_r}u+\rho(r)\right),\tag{18}$$

whenever $\Omega_r\subset\Omega^1$. As Ω^1 is arbitrary, we obtain as a consequence.

Theorem 3. *Let $u\in W^1_{m>0,loc}(\Omega)$ be a nonnegative, locally bounded solution of (1), (2) with structure (5)-(8). Then u is locally Hölder continuous in Ω.*

Now we consider problem (1), (2) with T assumed to be a nonnegative Radon measure μ and some conditions on μ that will ensure that the weak solution is locally Hölder continuous.

Theorem 4. *Let μ be a nonnegative Radon measure on Ω, and let $u\in W^1_{m,0,loc}(\Omega)$, $1<m<n$, be a nonnegative, bounded solution of (4), (2) with structure (5)-(8). Then*

$$r^{m-n}\int\limits_{\Omega_r}|\nabla u|^m\,dx\leq C\left[\lambda(r/2)-\lambda(r)+\rho(r)\right]^{m-1},\tag{19}$$

where $\lambda(r)=\inf\limits_{\Omega_r}u$.

Proof. Since μ is a nonnegative measure, u is therefore a weak supersolution of the quation (4) with $\mu=0$. Consider arbitrary weak supersolution υ. It follows from the energy estimate in Lemma 1 and the techniques of [1], [2] that

$$\int\limits_{\Omega_r}\upsilon^{-\beta-1}\,|\nabla\upsilon|^m\,\eta^m dx\leq C\left(1+\beta^{-m}\right)\int\limits_{\Omega_r}\upsilon^{m-\beta-1}\left\{\eta^m+|\nabla\eta|^m\right\}dx,\tag{20}$$

whenever $\beta>0$ and η is a cut-off function. Let θ be a positive number such that $1<(1-\theta)\,m<n/(n-m)$. Then

$$\int\limits_{\Omega_{r/2}}|\nabla\upsilon|^{m-1}\,dx\leq$$

$$\leq\left(\int\limits_{\Omega_{r/2}}\left(\upsilon^{-(1-\theta)m}\,|\nabla\upsilon|^m\right)dx\right)^{(m-1)/m}\left(\int\limits_{\Omega_{r/2}}\left(\upsilon^{(1-\theta)(m-1)m}\right)dx\right)^{1/m}.\tag{21}$$

It follows from (20) that the first factor is bounded by $Cr^{1-m}\left(\int\limits_{\Omega_r} v^{\theta m}dx\right)^{(m-1)/m}$ provided η is taken so that $\eta = 1$ $B_{r/2}$ with support contained in B_r. Then by theorem 2 we have

$$r^{m-n-1}\int\limits_{\Omega_{r/2}} |\nabla v|^{m-1}\,dx \le C\left(\lambda(r/2)+\rho(r)\right)^{(m-1)}.$$

By lemma 1 we have

$$\int\limits_{\Omega_{r/4}} |\nabla v|^{m}\,dx \le C\left\{\left(\lambda(r/2)+\rho(r)\right)^{(m-1)} r^{n-m} + r^{n-m}\rho^{m-1}(r)\right\}.$$

We establish the theorem, let $v = u - \inf\limits_{\Omega_r} u$.

Theorem is proved.

This estimates leads immediately to the following necessary condition for a solution to be Hōlder continuous.

Theorem 5. *If $u \in W^1_{m,0,loc}(\Omega)$, $1 < m < n$, is a nonnegative, bounded solution of (4), (2), that is locally Hölder continuous on Ω, then there exists $\varepsilon > 0$ such that*

$$\mu\left[B_r(x)\right] \le Cr^{n-m+\varepsilon},$$

whenever $B_{4r} \subset \Omega$, $x \in \Omega_r(x_0)$. Here C depends on the structure of the equation while ε depends also on the Hōlder exponent of u.

Proof. Let φ be a smooth function that is 1 on $B_r(x)$ and with support contained in $B_{2r}(x)$. Then appealing to the structure (5)-(8), and the previous theorem, we obtain

$$\mu\left(B_r(x)\right) \le \int\limits_{\Omega_{2r}} \varphi d\mu \le C_1 r^{-1}\int\limits_{\Omega_{2r}} \left|a_i\left(x,u,u_x\right)\right| dx + C_1\int\limits_{\Omega_{2r}} \left|a\left(x,u,u_x\right)\right| dx \le$$

$$\le C_1 r^{-1}\left(\int\limits_{\Omega_{2r}} |\nabla u|^{m-1}\,dx + \int\limits_{\Omega_{2r}} |u|^{m-1}\,dx + \int\limits_{\Omega_{2r}} |g_0|(x)dx\right) +$$

$$+C_1\left(\int\limits_{\Omega_{2r}} |u|^{m-1}\,dx + \int\limits_{\Omega_{2r}} k_0(x)dx\right) \le$$

later estimates we have

$$\le C_2\left[r^{n-m}\left(\lambda(4r)-\lambda(8r)+\rho(r)\right)^{m-1} + r^{n-m}\left(\lambda(4r)-\lambda(8r)+\rho(r)\right)^{m-1} + r^{n-m}\rho^{m-1}(r)\right].$$

Now using the assumption that u is locally Hōlder continuous and that $\rho(r)$ is bounded by a positive power of r, the conclusion follows.

Now to determine conditions under which the converse of theorem holds. First we prove one locally bounded of solutions of (4), (2). For this, we require the results of Adams [3].

Theorem (Adams). *Let μ be a nonnegative Radon measure supported in Ω such that for all $x \in \Omega$ and $0 < r < \infty$, there is a constant M with the property that $\mu\left(B_r(x)\right) \leq Mr^{\alpha}$, where $\alpha = q\left(n/m - 1\right)$, $1 < m < q < \infty, m < n$. If $u \in W^1_{m,0}(\Omega)$, then*

$$\left(\int\limits_{\Omega_r} |u|^q \, d\mu\right)^{1/q} \leq CM^{\frac{1}{q}} \left\|\nabla u\right\|_p .$$

Theorem 6. *If $u \in W^1_{m,0,loc}(\Omega), 1 < m < n$ is a weak solution of (4), (2) with structure (5)-(8), where μ is a nonnegative Radon measure supported in Ω_r with the property that for some $\varepsilon > 0, \mu[B_r(x)] \leq M \cdot r^{n-m+\varepsilon}$ for all $x \in \Omega_r$ and $0 < r < \infty$. For $0 < \sigma < 1$ and $\Omega_\rho \subset \Omega_r$, there exists $\alpha, \beta > 0$ such that*

$$\sup_{\Omega_{\sigma\rho}} u \leq C\rho^{-1/\alpha} \left(\int\limits_{\Omega_\rho} |u|^m \, dx + \int\limits_{\Omega_\rho} |u|^m \, d\mu\right)^{1/m} + C\rho^{n\beta}.$$

Proof. Determined $k_i = k(1 - 2^{-i}), i = 0, 1, 2, ...,$ and $r_i = \sigma\rho + 2^{-i}\rho(1-\sigma)$, $\bar{r}_i = \dfrac{1}{2}\left(r_i + r_{i+1}\right) = \sigma\rho + \dfrac{3}{4} 2^{-i}, i = 0, 1, 2, ...,$ the corresponding balls $B_{r_i}, B_{\bar{r}_i}$, $(r_i - r_{i+1})^{-1} = \dfrac{2^{i+1}}{\rho(1-\sigma)}$. Denote by ξ_i the cut-off function with support contained in $B_{\bar{r}_i}$ such that $\xi_i = 1$ on $B_{r_{i+1}}, |\nabla \xi_i| \leq 2^{i+2}/\rho(1-\sigma)$. Let $A_i = B_{r_i} \cap \{(u - k_{i+1})^+ > 0\}$. Substitute the test function $\varphi = \xi_i^m (u - k_{i+1})^+$ and employed Lemma 1 we obtain

$$\int\limits_{\Omega_{r_{i+1}}} \left|\nabla \left(u - k_{i+1}\right)^+\right|^m dx \leq C_1 \left[(1-\sigma)^{-m} \rho^{-m} \int\limits_{\Omega_{\bar{r}_i}} \left|\left(u - k_{i+1}\right)^+\right|^m dx + \right.$$

$$\left. + \int\limits_{\Omega_{\bar{r}_i}} \left|\left(u - k_{i+1}\right)^+\right|^m d\mu + \left(k^m + \lambda\right) |A_i| + \mu(A_i) \right],$$

because

$$\int\limits_{\Omega_{\bar{r}_i}} \varphi d\mu = \int\limits_{\Omega_{\bar{r}_i}} \xi_i^m \left(u - k_{i+1}\right)^+ d\mu \leq \int\limits_{\Omega_{\bar{r}_i}} \left(u - k_{i+1}\right)^+ d\mu \leq$$

$$\leq C_2 \left(\int\limits_{\Omega_{\bar{r}_i}} \left|\left(u - k_{i+1}\right)^+\right|^m d\mu + \mu(A_i) \right).$$

If q is defined by $n - m + \varepsilon = (q/m)\,(n-m)$, then the previous results of Adams implies that $\varphi \in L^q(d\mu)$, here $q/m > 1$.

Now

$$\int\limits_{\Omega_{r_{i+1}}} \left|\left(u - k_{i+1}\right)^+\right|^m d\mu \leq \int\limits_{\Omega_{\bar{r}_i}} \left|\left(u - k_{i+1}\right)^+ \xi_i\right|^m d\mu \leq$$

$$\leq \left(\int\limits_{\Omega_{\bar{r}_i}} \left|(u-k_{i+1})^+ \xi_i\right|^q d\mu \right)^{m/q} \mu(A_i)^{1-m/q} \leq$$

$$\leq C_3 \left(\int\limits_{\Omega_{\bar{r}_i}} \left|\nabla (u-k_{i+1})^+\right|^m dx + \int\limits_{\Omega_{\bar{r}_i}} \left|(u-k_{i+1})^+\right|^m |\nabla \xi_i|^m dx \right) \mu(A_i)^{1-m/q} \leq$$

$$\leq C_3 2^{im} (1-\sigma)^{-m} \rho^{-m} \left(\int\limits_{\Omega r_i} \left|(u-k_i)^+\right|^m dx + \int\limits_{\Omega r_i} \left|(u-k_i)^+\right|^m d\mu + \right.$$

$$\left. + (k^m+\lambda) |A_i| + \mu(A_i) \right) \mu(A_i)^{1-m/q}. \tag{22}$$

We have

$$\mu(A_i) \leq C_4^{mi} k^{-m} \int\limits_{\Omega_{r_i}} \left|(u-k_i)^+\right|^m d\mu \leq C_4^{mi} J_i,$$

$$|A_i| \leq C_4^{mi} k^{-m} \int\limits_{\Omega_{r_i}} \left|(u-k_i)^+\right|^m dx \leq C_4^{mi} J_i,$$

where

$$J_i = k^{-m} \int\limits_{\Omega_{r_i}} \left|(u-k_i)^+\right|^m dx + k^{-m} \int\limits_{\Omega_{r_i}} \left|(u-k_i)^+\right|^m d\mu.$$

If β is chosen so that $0<\beta<1-m/q$ from (22) it follows that

$$k^{-m} \int\limits_{\Omega_{r_{i+1}}} \left|(u-k_{i+1})^+\right|^m d\mu \leq C_5 C_4^{mi} (1-\sigma)^{-m} \rho^{-m} \left[J_i \left(C_4^{mi} J_i\right)^{1-m/q} + \right.$$

$$\left. + (k^m+\lambda) r^{n\beta} k^{-m} \left(C_4^{mi} J_i\right)^{1-\beta} \left(C_4^{mi} J_i\right)^{1-m/q} + \left(C_4^{mi} J_i\right) \left(C_4^{mi} J_i\right)^{1-m/q} \right. .$$

If k is chosen so that $k^m \geq \lambda \rho^{n\beta}$ and $k \geq \|u\|_{L_p(B_{r_0})}$, we have for some $\alpha>0$,

$$k^{-m} \int\limits_{\Omega_{r_{i+1}}} \left|(u-k_{i+1})^+\right|^m d\mu \leq C_5 C_4^{mi} (1-\sigma)^{-m} \rho^{-m} J_i^{1+\alpha}. \tag{23}$$

Analogously we estimate $\int\limits_{\Omega_{r_{i+1}}} \left|(u-k_{i+1})^+\right|^m dx$ under $q<nm/(n-m)$. Hence

$$k^{-m} \int\limits_{\Omega_{r_{i+1}}} \left|(u-k_{i+1})^+\right|^m dx \leq C_5 C_4^{mi} (1-\sigma)^{-m} \rho^{-m} J_i^{1+\alpha}. \tag{24}$$

From (23) and (24) we obtain

$$J_{i+1} \leq C_5 C_4^{mi} (1-\sigma)^{-m} \rho^{-m} J_i^{1+\alpha},$$

here $C_4^m>1$, $k^m \geq \lambda \rho^{n\beta}$, $k \geq \|u\|_{L_p(B_0)}$. From [4] implies that $J_i \to 0$ provided

$$J_0 \leq C_6 [(1-\sigma)\rho]^{m/\alpha},$$

or

$$J_0 = k^{-m} \int\limits_{\Omega_\rho} |u|^m \, dx + k^{-m} \int\limits_{\Omega_\rho} |u|^m \, d\mu \le C_6 [(1-\sigma)\,\rho]^{m/\alpha}.$$

Theorem is proved.

Later we assume that the operator $a_i\,(x,u,p)$ is strongly monotonic in Ω.

$$\left(a_i(x,u,p) - a_i(x,u,p')\right)\left(p-p'\right) \ge C\left|p-p'\right|^m, \tag{25}$$

for all $x \in \Omega,\ u \in R,\ p,\ p' \in R^n$..

Theorem 7. *Let $u \in W^1_{m,0,loc},(\Omega), 1 < m < n$, be a weak solution of (4), (2) with μ is nonnegative measure with the property $\mu\,(B_r(x)) \le c r^{n-m+\varepsilon}$ for all balls $B_r(x)$ where $B_{2r}(x) \subset \Omega_r,\ x \in \Omega_r$. We assume structure (5)-(8) and (25), then u is locally Hōlder continuous on Ω.*

Proof. By theorem 6 the solution u is bounded in Ω_r, because we may consider $a_i(x,u,p)$ as $a_i(x,p)$ and normalize the structure we obtain

$$a_i(x,p)p_i \ge |p|^m - \nu_0, |a_i\,(x,p)| \le |p|^{m-1} + \nu_1,$$

where $\nu_1 \in L^{q_1}(\Omega_r), \nu_0 \in L^{q_2}(\Omega_r)$ with $q_1 > n/(m-1), q_2 > n/m$..

Now let $\upsilon \in W^1_{m,0}(\Omega_r)$ be defined as a solution of

$$\frac{d}{dx_i} a_i\,(x, \upsilon_{x_i}) = 0,$$

where $u - \upsilon \in W^1_{m,0}(\Omega_r)$. The existence of υ is provided in [4] and because u is bounded on Ω_r we may appeal [4] to conclude that υ is bounded on Ω_r. Using the strong monotonocity of a_i and that $u - \upsilon$ is a test function, we obtain

$$\int\limits_{\Omega_r} |\nabla (u-\upsilon)|^m \, dx \le C_1 \left(\int\limits_{\Omega_i} \left(a_i\,(x,u,u_{x_i}) - a_i\,(x,u,\upsilon_{x_i})\right)(u-\upsilon)_{x_i} + \right.$$

$$\left. + \int\limits_{\Omega_r} a\,(x,u,u_{x_i})\,(u-\upsilon)\,dx + \int\limits_{\Omega_r} (u-\upsilon)\,d\mu \right) \le C_2 \left(\int\limits_{\Omega_r} |\nabla u|^{m-1} |u-\upsilon| \, dx + \right.$$

$$\left. + \int\limits_{\Omega_r} k_0\,(u-\upsilon)\,dx + r^{n-m+\varepsilon} \right). \tag{26}$$

We applied Young and Poincare inequality obtain

$$\int\limits_{\Omega_r} |\nabla u|^{m-1} |u-\upsilon| \, dx \le \|\nabla u\|^{m-1}_{L_m(\Omega_r)} \|u-\upsilon\|_{L_m(\Omega_r)} \le$$

$$\|\nabla u\|^{m-1}_{L_m(\Omega_r)} r \, \|\nabla (u-\upsilon)\|_{L_m(\Omega_r)} \le$$

$$\leq C_3(\alpha)r^{m'}\int\limits_{\Omega_r}|\nabla u|^m\,dx+\alpha\int\limits_{\Omega_r}|\nabla(u-v)|^m\,dx, \tag{27}$$

where C_3 depends on the structure, $\alpha>0$ for any real number. Consequently, (26) can be written

$$\int\limits_{\Omega_r}|\nabla(u-v)|^m\,dx\leq C_4\left(r^{m'}\int\limits_{\Omega_r}|\nabla u|^m\,dx+\int\limits_{\Omega_r}k_0\,(u-v)\,dx+r^{n-m+\varepsilon}\right). \tag{28}$$

Since $(u-v)$ the local boundedness, implies that

$$\int\limits_{\Omega_r}k_0(u-v)dx\leq C_5r^{n-m+\varepsilon},$$

where ε' depends on $q>n/m$. Thus, we obtain from (28)

$$\int\limits_{\Omega_r}|\nabla(u-v)|^m\,dx\leq C_6r^{m'}\int\limits_{\Omega_r}|\nabla u|^m\,dx+C_7r^{n-m+\varepsilon} \tag{29}$$

for some $\varepsilon>0$. It follows from the fundamental energy estimates in [5], lemma 1 that

$$\int\limits_{\Omega_\rho}|\nabla v|^m\,dx\leq C\left(\frac{\rho}{r}\right)^{n-m+\varepsilon'}\int\limits_{\Omega_\rho}|\nabla v|^m\,dx, \tag{30}$$

for all $0<\rho<r$, and some $\varepsilon'>0$. Now we obtain a bound for $\int\limits_{\Omega_\rho}|\nabla v|^m\,dx$. Since

$$\int\limits_{\Omega_\rho}a_i\left(x,v_{x_i}\right)(u-v)_{x_i}\,dx=\int\limits_{\Omega_r}a_i\left(x,u,v_{x_i}\right)(u-v)_{x_i}\,dx=0,$$

it follows that

$$\int\limits_{\Omega_r}|\nabla v|^m\,dx\leq C_8\left(\int\limits_{\Omega_r}|\nabla u|^m\,dx+r^{n-m+\varepsilon}\right). \tag{31}$$

Since $u=v+(u-v)$ and thus from (29), (30), (31) we obtain

$$\int\limits_{\Omega_r}|\nabla u|^m\,dx\leq\int\limits_{\Omega_r}|\nabla v|^m\,dx+\int\limits_{\Omega_r}|\nabla(u-v)|^m\,dx\leq$$

$$\leq\left[\left(\frac{\rho}{r}\right)^{n-m+\varepsilon'}+r^{m'}\right]\int\limits_{\Omega_r}|\nabla v|^m\,dx+r^{n-m+\varepsilon}.$$

Now employed the technique in [4], [6] to conclude that there exists $\varepsilon_0>0$ such that if

$$r^{m-n}\int\limits_{\Omega_r}|\nabla u|^m\,dx\leq\varepsilon_0, \tag{32}$$

then there exists $\varepsilon_0 > 0$ and constant C depending only on the structure such that

$$\rho^{m-n} \int_{\Omega_\rho} |\nabla u|^m \, dx \le C \left(\frac{\rho}{r} \right)^{\varepsilon}.$$

As in [4], [6] we conclude that u is locally Hōlder continuous in Ω_r. This conclusion is based on (32) which is know to hold at all points $x_0 \in \Omega_r$ by theorem 4.

Theorem 8. *Let $u \in W^1_{m,0,loc}(\Omega)$ be a weak solution of (1), (2) with structure (5)-(8) and $T \in W^{-1}_{m'+\varepsilon,loc}(\Omega)$ for some $\varepsilon > 0$. Then there $q > m$ such that $u \in W^1_{q,0,loc}(\Omega)$. Moreover,*

$$\left(\fint_{\Omega_{r/2}} |\nabla u|^q \, dx \right)^{1/q} \le C \left(\left(\fint_{\Omega_r} |\nabla u|^m \, dx \right)^{1/m} + \left(\fint_{\Omega_r} h^q dx \right)^{1/q} \right), \qquad (33)$$

where $h = \left(g_0^{m'} + |g|^{m'} \right)^{1/m}$, $div = T + k_0$.

Proof. Let $\bar{u}(r) = \fint_{\Omega_r} u\,(y)\,dy$ and define a test function $\varphi = \eta^m \left(u - \bar{u}(r) \right)$. We obtain

$$\int_{\Omega_r} \eta^m \, |\nabla u|^m \, dx \le$$

$$\le C_1 \left(\int_{\Omega_r} |\nabla u|^{m-1} |\nabla u| \left(u - \bar{u}(r) \right) \eta^{m-1} dx + \int_{\Omega_r} g_0 \left(u - \bar{u}(r) \right) \eta^{m-1} |\nabla \eta| \, dx + \right.$$

$$\left. + C_2 \int_{\Omega_r} \left(u - \bar{u}(r) \right) |\nabla u|^{m-1} \eta^m dx \right) + \left\langle div\ g, \left(u - \bar{u}(r) \right) \eta^m \right\rangle,$$

where $g \in L_{m'+\varepsilon}(\Omega)$. The last term replaced by $C_3 \int_{\Omega_r} |g| \left| \nabla \left[\left(u - \bar{u}(r) \right) \eta^m \right] \right| dx$.

For the conclusion see [4], [6].

Theorem (Hedberg, Wolf). *Let $m > 1$ and $km < n$. If μ is a Radon measure, then $\mu \in W^k_m(R^n)$ if and only if*

$$\int_{R^n} \int_0^1 \left(\frac{\mu[B_r(y)]}{r^{n-km}} \right)^{1/(m-1)} \frac{dr}{r} d\mu(y) < \infty.$$

As a consequence of theorem 8 and this theorem we obtain analogously estimate of type (33) for a weak solution of (4), (2).

Theorem 9. *Let $u \in W^1_{m,0,loc}(\Omega)$ be a nonnegative weak solution of (4), (2) with structure (5)-(8), where μ is a nonnegative measure supported in Ω with the property $\mu(B_r(x)) \le C r^{n-m+\varepsilon}$ for all balls $B_r(x)$, where $B_{2r}(x) \subset \Omega_r$, $x \in \Omega_r$.*

Then there exists $\alpha, \beta > 0$ and ε such that for $0 < r < 1$

$$\sup_{\Omega_{r/2}(x)} u \le C r^{-\alpha} \left[\left(\inf_{\Omega_{r/2}(x)} u + \rho(r) \right)^{m-1} \right] + C r^{\beta}, \ \text{if} \ 1 < m < 2,$$

$$\sup_{\Omega_{r/2}(x)} u \le Cr^{-\alpha}\left[\left(\inf_{\Omega_{r/2}(x)} u + \rho(r)\right)\right] + Cr^{\beta},\ \text{if}\ m \ge 2.$$

Proof. Choose $x \in \Omega_r$ and $0 < r < 1$ such that $B_{2r}(x) \subset \Omega_r$. With η smooth cut-off function, we substitute the test function $\varphi = u\eta^m$ to obtain

$$\int_{\Omega_r} ud\mu \le \int_{\Omega_r} \varphi d\mu = \int_{\Omega_r} \left[a_i(x,u,u_x)\varphi_{x_i} + a(x,u,u_x)\varphi\right] dx.$$

Using (5)-(8), that u is locally bounded and applying theorem 4, we have

$$\int_{\Omega_r} ud\mu \le \int_{\Omega_r} \varphi d\mu \le C\left[(\lambda(4r) - \lambda(8r) + \rho(r))^{m-1} r^{n-m} + \rho^{m-1}(r) r^{n-m}\right]. \qquad (34)$$

We use theorem 2 and the fact that u is locally bounded to obtain

$$\int_{\Omega_r} udx \le C\int_{\Omega_r} u^{m-1}dx \le Cr^n(\lambda(r/2) + \rho(r))^{m-1},$$

when $1 < m < 2\,(0 < m - 1 < n(m-1)/\,(m - m))$ and that

$$\int_{\Omega_r} udx \le Cr^n(\lambda(r/2) + \rho(r)),$$

when $m \ge 2$ $(n(m-1)/\,(n-m) > 1)$. These conclusions along with (34), theorem 6 and fact that by interpolation technique it is easy to see that the above estimates remains valid with m replaced by q for any $q > 0$, establish the result.

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