MATHEMATICS

Fuad A. ABDULLAYEV

ON ONE STRUCTURAL PROPERTY OF A DOUBLE SINGULAR INTEGRAL WITH A HILBERT KERNEL

Abstract

In this paper the scale of invariant Banach spaces is constructed for a double singular integral with a Hilbert kernel

$$\tilde{f}(x,y)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s,t)\, ctg\frac{x-s}{2} ctg\frac{y-t}{2} dsdt$$

in the language of partial and mixed smoothness of arbitrary fractional order.

Let us consider the following double singular integral (s.i.)

$$\tilde{f}(x,y)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s,t)\, ctg\frac{x-s}{2} ctg\frac{y-t}{2} dsdt, \tag{1}$$

where the density $f(s,t)$ is a 2π-periodic function on each of variables, continuous on $T^2=[-\pi,\pi]^2$. Note that integral (1) is understood in the following sense:

$$\tilde{f}(x,y)\overset{des}{=}\lim_{\substack{\varepsilon_1\to+0\\ \varepsilon_2\to+0}}\int_{T^2\setminus\Pi_{\varepsilon_1}(x)\times\pi_{\varepsilon_2}(y)} f(s,t)\, ctg\frac{x-s}{2} ctg\frac{y-t}{2} dsdt, \tag{2}$$

where for $\xi\in(0,\pi]$

$$\Pi_\varepsilon(\xi)=\begin{cases}[\xi-\varepsilon,\xi+\varepsilon], & if\quad \xi+\varepsilon\le\pi,\\ [\xi-\varepsilon,\pi]\cup[-\pi,-2\pi+\xi+\varepsilon], & if\quad \xi+\varepsilon>\pi,\end{cases}$$

and for $\xi\in[-\pi,0]$

$$\Pi_\varepsilon(\xi)=\begin{cases}[\xi-\varepsilon,\xi+\varepsilon], & if\quad \xi-\varepsilon\ge-\pi,\\ [-\pi,\xi+\varepsilon]\cup[2\pi+\xi-\varepsilon,\pi], & if\quad \xi-\varepsilon<-\pi\end{cases}$$

Following L.Cezari [1] we call $\tilde{f}(x,y)-$ a function conjugate to $f(x,y)$ with respect to set of variables x and y.

The integrals of form (1) are met at studying the conjugate double trigonometric series ([2]-[4]), in relations between limit values in the frame of boundary of bicylinder domain, and in the theory of singular integral equations [13].

The studying of s.i. (1) has been first began by L.Cezari [1]. He has proved that

$$\left|\tilde{f}(x_1,y_1)-\tilde{f}(x_2,y_2)\right|=0\left(|x_2-x_1|^\alpha\ln\frac{\pi}{|x_2-x_1|}+|y_2-y_1|^\alpha\ln\frac{\pi}{|y_2-y_1|}\right),$$

if

$$|f(x_1,y_1)-f(x_2,y_2)|=0\left(|x_2-x_1|^\alpha+|y_2-y_1|^\alpha\right),0<\alpha<1,$$

i.e. in smoothness logarithmical loss takes place . Moreover, I.E.Jack [2] showed that this estimation is exact by order, and consequently, the analogy of I.Plemel - I.I.Privalov theorem does not hold for s.i. (1). I.E.Jack [2] showed the possibility of obtaining the analogy of this theorem in the definite sense, by substituting the condition $f \in Lip\alpha$ by the other ones.

As is known the I.Plemel - I.I.Privalov theorem at $\alpha = 1$ is invalid and is substituted by the following statement.

Theorem (A.Zygmund). *If the continuous $\imath\pi$-periodic function $u(x)$ satisfies the condition*

$$|u(x+h) - 2u(x) + u(x-h)| = 0(h), h \to 0,$$

uniformly with respect to x, then the function

$$\tilde{u}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\xi)\, ctg\frac{\xi - x}{2} d\xi \tag{3}$$

satisfies this condition.

I.E.Jock showed that this theorem is not transferred to s.i. (1).

In the paper [3] the following estimate is established by S.B.Stechkin and N.K.Bari for s.i. (3)

$$\forall k, l \in N \quad \omega_{\tilde{u}}^{k}(\delta) \le const\left(\int_0^{\delta} \frac{\omega_u^l(s)}{s} ds + \delta^k \int_{\delta}^{\pi} \frac{\omega_u^l(s)}{s^{k+1}} ds\right), \tag{4}$$

on the basis of which the necessary and sufficient condition is obtained on φ for

$$\omega_u^k(\delta) = O(\varphi(\delta)) \iff \omega_{\tilde{u}}^k(\delta) = O(\varphi(\delta)), \tag{5}$$

where $\varphi \in \Phi^k = \left\{\varphi \in \Phi | 0 < t_1 < t_2 \le \pi \iff t_1^k \varphi(t_2) \le C t_2^k \varphi(t_1)\right\}$ and Φ is a class of functions $\varphi(t)$, determined on $[0,\pi]$ and having the properties: $\varphi(t)$ is continuous on $[0,\pi]$, $\varphi(t)$ monotonically increases, $\varphi(t) \ne 0$ for any $t \in (0,\pi]$ and $\varphi(0) = 0$ (3).

In the paper [11] for arbitrary functional $r \ge 1$ the following estimate is obtained

$$\omega_{\tilde{u}}^{r}(\delta) \le const\left(\int_0^{\delta} \frac{\omega_u^r(s)}{s} ds + \delta^{rk} \int_{\delta}^{\pi} \frac{\omega_u^r(s)}{s^{r+1}} ds + \delta^r ||u||_C\right), \tag{6}$$

on the basis of which the sufficient condition on φ for

$$\omega_u^r(\delta) = O(\varphi(\delta)) \iff \omega_{\tilde{u}}^r(\delta) = O(\varphi(\delta)),$$

where $\varphi \in \Phi^r$ is found.

Later on the properties of s.i. (1) were studied in the papers [4-5]. The results of these papers show that for obtaining the analysis of I.Plemel-I.I.Privalov, A.Zigmund, N.K.Bari and S.B.Stechkin theorems for s.i. (1), it is insufficient to work only in language of partial and mixed modules of smoothness. In language of partial and mixed module of continuity the analogy of the I.Plemel-I.I.Privalov theorems for s.i. (1) has been obtained in the papers [4-5]. And in paper [10] for s.i. (1) was constructed the analogy of N.K.Bari and S.B.Stechkin theoerm.

In this paper the results of the paper [11] obtained for s.i. (3) is transferred to s.i. (1).

Denote by C_{T^2} a space of functions continuous on T^2, 2π-periodic on each of variables with the norm

$$\|f\|_C = \max_{(x,y)\in T^2} |f(x,y)| \ .$$

Let us introduce the following notation

$\omega_f^{r,\rho}(\delta,\eta) = \sup\limits_{|h_1|\le\delta,\ h_2\le\eta} \left\|\Delta_{h_1,h_2}^{r,\rho} f(x,x)\right\|_C$, $\delta,\eta\in(0,\pi]$ is mixed smoothness module of $r>0$-order by the first argument and of $\rho>0$-order by the second argument;

$\omega_f^{r,0}(\delta) = \sup\limits_{|h|\le\delta} \left\|\Delta_h^{r,0} f(x,x)\right\|_C$, $\delta\in(0,\pi]$ is specific smoothness module of the r-order by the first argument;

$\omega_f^{0,\rho}(\delta) = \sup\limits_{|h|\le\delta} \left\|\Delta_h^{0,\rho} f(x,x)\right\|_C$, $\delta\in(0,\pi]$ is specific smoothness module of the ρ-order by the second argument, where r,ρ are arbitrary positive numbers and

$$\Delta_h^{r,0} f(x,y) = \exp(\pi r i) \sum_{j=0}^{\infty} A_j^{-r-1} f(x+jh,y)\,, \tag{7}$$

$$\Delta_h^{0,\rho} f(x,y) = \exp(\pi \rho i) \sum_{j=0}^{\infty} A_j^{-\rho-1} f(x,y+jh)\,, \tag{8}$$

$$\Delta_{h_1,h_2}^{r,\rho} f(x,y) = \exp(\pi(r+\rho)i) \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} A_j^{-r-1} A_m^{-\rho-1} f(x+jh_1,y+mh_2)\,, \tag{9}$$

where we assume that $\Delta_{h_1,h_2}^{0,0} f(x,y) = f(x,y)$.

In formulae (7)-(9) the coefficients A_j^{-r-1} and $A_j^{-\rho-1}$ are determined from the relations

$$(1-x)^r = \sum_{j=0}^{\infty} A_j^{-r-1} x^j, \qquad (1-x)^\rho = \sum_{j=0}^{\infty} A_j^{-\rho-1} x^j. \tag{10}$$

It is obvious that for $r>0$, $\rho>0$

$$\sum_{j=0}^{\infty} A_j^{-r-1} = \sum_{j=0}^{\infty} A_j^{-\rho-1} = 0\ .$$

The numbers A_n^α are called α-order Cezari numbers. For them there exists the explicit representation [7]

$$A_n^\alpha = \frac{(\alpha+1)\ldots(\alpha+n)}{n!} = \binom{n+\alpha}{n} = O(n^\alpha) \qquad (\alpha\ne -1,-2,\ldots) \tag{11}$$

If $r>0$ and $\rho>0$ are entire, then differences (7)-(9) turn into ordinary differences of entire order, since at $j\ge r+1$, $m\ge\rho+1$ $A_j^{-r-1} = A_m^{-p-1} = 0$.

From (11) it follows that

$$\sum_{j=0}^{\infty} \left|A_j^{-r-1}\right| < \infty\ , \qquad \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \left|A_j^{-r-1} A_m^{-\rho-1}\right| < \infty$$

Hence, in particular, it follows that series (7)-(9) converge in C_{T^2}.

Analogously, to [11] we can prove the following theorem for the characteristics $\omega_f^{r,\rho}(\delta,\eta)$, $\omega_f^{r,0}(\delta)$ and $\omega_f^{0,\rho}(r)$.

Theorem 1. (Analogy of Marchoud inequality). *Let $f \in C_{T^2}$. Then for any $r > 0,\ r_1 \geq 0,\ \rho > 0,\ 0 < \delta \leq \frac{\pi}{r}$ and $r \in (0,\pi]$ the inequaltiy*

$$\omega_f^{r_1,\rho}(\delta,\eta) \leq C\delta^{r_1}\left(\int\limits_{\delta}^{\pi/2} \frac{\omega_f^{r+1,\rho}(t,\eta)}{t^{r_1+1}}dt + \omega_f^{0,\rho}(r)\right) \tag{12}$$

is valid.

Theorem 2. *Let $f \in C_{T^2}$. Then for any $r > 0, \rho > 0, \rho_1 \geq 1, 0 < \eta \leq \pi/\rho$ and $\delta \in (0.\pi]$ the inequality*

$$\omega_f^{r,\rho_1}(\delta,\eta) \leq C\delta^{\rho_1}\left(\int\limits_{\eta}^{\pi/2} \frac{\omega_f^{r,\rho+1}(\delta,t)}{t^{\rho_1+1}}dt + \omega_f^{r,0}(\delta)\right) \tag{13}$$

is valid.

Theorem 3. *Let $f \in C_{T^2}$. Then for any $r > 0,\ r_1 \geq 1$ and $0 < \delta < \frac{\pi}{r}$ the inequality*

$$\omega_f^{r_1,0}(\delta) \leq C\delta^{r_1}\left(\int\limits_{\delta}^{\pi} \frac{\omega_f^{r+1,0}(t)}{t^{r_1+1}}dt + \|f\|_C\right) \tag{14}$$

holds .

Theorem 4. *Let $f \in C_{T^2}$. Then for any $\rho > 0,\ \rho_1 \geq 1$ and $0 < \eta < \frac{\pi}{\rho}$ the inequality*

$$\omega_f^{0,\rho_1}(\eta) \leq C\eta^{\rho_1}\left(\int\limits_{\eta}^{\pi} \frac{\omega_f^{0,\rho+1}(t)}{t^{\rho_1+1}}dt + \|f\|_C\right) \tag{15}$$

is valid.

Consider the double singular operator

$$(Sf)(x,y) \equiv \tilde{f}(x,y) = \frac{1}{4\pi^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} f(s,t)\, ctg\frac{s-x}{2} ctg\frac{t-y}{2} dsdt \tag{16}$$

The following theorem holds.

Theorem 5. *Let $f \in C_{T^2},\ r \geq 1,\ \rho \geq 1$ and*

$$\int\limits_0^{\pi}\int\limits_0^{\pi} s^{-1}t^{-1}\omega_f^{r,\rho}(s,t)\,dsdt < \infty,\ \int\limits_0^{\pi} s^{-1}\omega_f^{r,0}(s)\,ds < \infty,\ \int\limits_0^{\pi} t^{-1}\omega_f^{0,\rho}(t)\,dt < \infty. \tag{17}$$

Then the estimation

$$\omega_{\tilde{f}}^{r,0}(\delta,\eta) \leq C\left(\int\limits_0^{\delta}\int\limits_0^{\eta} \frac{\omega_f^{r,\rho}(s,t)}{st}dsdt + \delta^r\int\limits_{\delta}^{\pi}\int\limits_0^{\eta} \frac{\omega_f^{r,\rho}(s,t)}{s^{r+1}t}dsdt + \right.$$

$$+\eta^{\rho}\int_{0}^{\delta}\int_{\eta}^{\pi}\frac{\omega_f^{r,\rho}(s,t)}{st^{\rho+1}}dsdt+\delta^{r}\eta^{\rho}\int_{\delta}^{\pi}\int_{r}^{\eta}\frac{\omega_f^{r,\rho}(s,t)}{s^{r+1}t^{\rho+1}}dsdt+$$

$$+\delta^{r}\int_{0}^{\eta}\frac{\omega_f^{0,\rho}(t)}{t}dt+\eta^{\rho}\int_{0}^{\delta}\frac{\omega_f^{r,0}(s)}{s}ds+\delta^{r}\eta^{\rho}\int_{\delta}^{\pi}\frac{\omega_f^{r,0}(s)}{s^{r+1}}ds+$$

$$\left.+\delta^{r}\eta^{s}\int_{\eta}^{\pi}\frac{\omega_f^{0,\rho}(t)}{t^{\rho+1}}dt+\delta^{r}\eta^{\rho}\left\|f\right\|_C\right) \tag{18}$$

is valid.

Proof. For whole $r\geq 1,\ \rho\geq 1$ estimation (18) is known [10]. Let $r\geq 1,\ \rho\geq 1$ be any non-integer numbers.

By theorem 1 we have

$$\omega_{\tilde f}^{r,\rho}(\delta,r)\leq C\delta^{r}\left(\int_{\delta}^{\pi}\frac{\omega_{\tilde f}^{[r]+1,\rho}(s,\eta)}{s^{r+1}}ds+\omega_{\tilde f}^{0,\rho}(\eta)\right). \tag{19}$$

We apply theorem 4 to $\omega_{\tilde f}^{0,\rho}(\eta)$. Then we obtain

$$\omega_{\tilde f}^{0,\rho}(\eta)\leq C\eta^{\rho}\left(\int_{\eta}^{\pi}\frac{\omega_{\tilde f}^{0,[\rho]+1}(t)}{t^{\rho+1}}dt+\left\|\tilde f\right\|_C\right). \tag{20}$$

In the paper [10] for $\omega_{\tilde f}^{0,[\rho]+1}(t)$ the estimate

$$\omega_{\tilde f}^{0,[\rho]+1}(t)\leq C\left(\int_{0}^{\pi}\int_{0}^{t}\frac{\omega_f^{[r]+1,[\rho]+1}(s,\tau)}{s\tau}dsd\tau+t^{[\rho]+1}\int_{0}^{\pi}\int_{t}^{\pi}\frac{\omega_f^{[r]+1,[\rho]+1}(s,\tau)}{s\tau^{[\rho]+2}}dsd\tau+\right.$$

$$+t^{[\rho]+1}\int_{\delta}^{\pi}\frac{\omega_f^{[r]+1,0}(s)}{s}ds+\int_{0}^{t}\frac{\omega_f^{0,[\rho]+1}(\tau)}{\tau}d\tau+$$

$$+t^{[\rho]+1}\int_{t}^{\pi}\frac{\omega_f^{0,[\rho]+1}(\tau)}{\tau^{[\rho]+2}}d\tau+t^{[\rho]+1}\left\|f\right\|_C \tag{21}$$

is established.

We put estimate (21) in (20) and estimate each of obtained addends

$$J_1=\eta^{\rho}\int_{\eta}^{\pi}\frac{t^{[\rho]+1}\left\|f\right\|_C}{t^{\rho+1}}dt\leq C\eta^{\rho}\left\|f\right\|_C . \tag{22}$$

$$J_2=\eta^{\rho}\int_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}\int_{0}^{\pi}\int_{0}^{t}\frac{\omega_f^{[r]+1,[\rho]+1}(s,\tau)}{s\tau}dsd\tau=$$

$$=\eta^{\rho}\int\limits_{0}^{\pi}\frac{ds}{s}\int\limits_{0}^{\eta}\frac{\omega_{f}^{[r]+1,[\rho]+1}(s,\tau)}{\tau}d\tau\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}+$$

$$+\eta^{\rho}\int\limits_{0}^{\pi}\frac{ds}{s}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{[r]+1,[\rho]+1}(s,\tau)}{\tau}d\tau\int\limits_{\tau}^{\pi}\frac{dt}{t^{\rho+1}}\leq$$

$$\leq\int\limits_{0}^{\pi}\int\limits_{0}^{\eta}\frac{\omega_{f}^{r,\rho}(s,\tau)}{st}dsdt+\eta^{\rho}\int\limits_{0}^{\pi}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{r,\rho}(s,t)}{st^{\pi+1}}dsdt\leq$$

$$\leq\int\limits_{0}^{\eta}\frac{\omega_{f}^{0,\rho}(t)}{t}dt+\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{0,\rho}(t)}{t^{\rho+1}}dt\ . \tag{23}$$

$$J_{3}=\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}t^{[\rho]+1}\int\limits_{0}^{\pi}\int\limits_{t}^{\pi}\frac{\omega_{f}^{[r]+1,[\rho]+1}(s,\tau)}{s\tau^{[\rho]+2}}dsd\tau=$$

$$=\eta^{\rho}\int\limits_{0}^{\pi}\frac{ds}{s}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{[r]+1,[\rho]+1}(s,\tau)}{\tau^{[\rho]+2}}d\tau\int\limits_{\eta}^{\pi}t^{[\rho]-\rho}dt\leq$$

$$\leq C\eta^{\rho}\int\limits_{0}^{\pi}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{r,\rho}(s,t)}{st^{\rho+1}}dsdt\leq C\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{0,\rho}(t)}{t^{\rho+1}}dt\ . \tag{24}$$

$$J_{4}=\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}t^{[\rho]+1}\int\limits_{0}^{\pi}\frac{\omega_{f}^{[r]+1,0}(s)}{s}ds\leq C\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{0,\rho}(t)}{t^{\rho+1}}dt\ . \tag{25}$$

$$J_{5}=\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}\int\limits_{0}^{t}\frac{\omega_{f}^{0,[\rho]+1}(\tau)}{\tau}d\tau=\eta^{\rho}\int\limits_{0}^{\eta}\frac{\omega_{f}^{0,[\rho]+1}(\tau)}{\tau}d\tau\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}+$$

$$+\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{0,[\rho]+1}(\tau)}{\tau}d\tau\int\limits_{\tau}^{\pi}\frac{dt}{t^{\rho+1}}\leq C\left(\int\limits_{0}^{\eta}\frac{\omega_{f}^{0,\rho}(t)}{t}dt+\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{0,\rho}(t)}{t^{\rho+1}}dt\right). \tag{26}$$

Analogously to J_3 we obtain

$$J_{6}=\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}t^{[\rho]+1}\int\limits_{t}^{\pi}\frac{\omega_{f}^{0,[\rho]+1}(\tau)}{\tau^{[\rho]+2}}d\tau\leq C\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{0,\rho}(t)}{t^{\rho+1}}dt\ . \tag{27}$$

Collecting all the obtained estimates, for $\omega_{\tilde{f}}^{0,\rho}(\eta)$ we find

$$\omega_{\tilde{f}}^{0,\rho}(\eta)\leq C\left(\int\limits_{0}^{\eta}\frac{\omega_{f}^{0,\rho}(t)}{t}dt+\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{0,\rho}(t)}{t^{\rho+1}}dt+\eta^{\rho}\left\|f\right\|_{C}\right). \tag{28}$$

We now estimate $\omega_{\tilde{f}}^{[r]+1,\rho}(s,\eta)$ in (19).

Again by theorem 1 we have

$$\omega_{\tilde{f}}^{[r]+1,\rho}(s,\eta) \le C\eta^{\rho}\left(\int\limits_{\eta}^{\pi}\frac{\omega_{\tilde{f}}^{[r]+1,[\rho]+1}(s,t)}{t^{\rho+1}}dt+\left\|\tilde{f}\right\|_{C}\right). \tag{29}$$

In the paper [10] the estimation

$$\omega_{\tilde{f}}^{[r]+1,[\rho]+1}(s,t)\le$$

$$\le C\left(\int\limits_{0}^{s}\int\limits_{0}^{t}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi\tau}d\xi d\tau+s^{[r]+1}\int\limits_{s}^{\pi}\int\limits_{0}^{t}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi^{[r]+2}\tau}d\xi d\tau+\right.$$

$$+t^{[\rho]+1}\int\limits_{0}^{s}\int\limits_{t}^{\pi}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi\tau^{[\rho]+2}}d\xi d\tau+s^{[r]+1}t^{[\rho]+1}\int\limits_{s}^{\pi}\int\limits_{t}^{\pi}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi^{[r]+2}\tau^{[\rho]+2}}d\xi d\tau+$$

$$+s^{[r]+1}\int\limits_{0}^{t}\frac{\omega_{f}^{0,[\rho]+1}(\tau)}{\tau}d\tau+t^{[\rho]+1}\int\limits_{0}^{s}\frac{\omega_{f}^{[r]+1}(\xi)}{\xi}d\xi+s^{[r]+1}t^{[\rho]+1}\int\limits_{s}^{\pi}\frac{\omega_{f}^{[r]+1,0}(\xi)}{\xi^{[r]+2}}d\xi+$$

$$\left.+s^{[r]+1}t^{[\rho]+1}\int\limits_{t}^{\pi}\frac{\omega_{f}^{0,[\rho]+1}(\tau)}{\tau^{[\rho]+2}}d\tau+s^{[r]+1}t^{[\rho]+1}\left\|f\right\|_{C}\right) \tag{30}$$

is obtained.

We put estimate (30) in (29) and estimate the obtained integrals

$$J_7=\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}\int\limits_{0}^{s}\int\limits_{0}^{t}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi\tau}d\xi d\tau=$$

$$=\eta^{\rho}\int\limits_{0}^{s}\frac{d\xi}{\xi}\int\limits_{0}^{\eta}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\tau}d\tau\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}+$$

$$+\eta^{\rho}\int\limits_{0}^{s}\frac{d\xi}{\xi}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\tau}d\tau\int\limits_{\tau}^{\pi}\frac{dt}{t^{\rho+1}}\le$$

$$\le C\left(\int\limits_{0}^{s}\int\limits_{0}^{\eta}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi\tau}d\xi d\tau+\eta^{\rho}\int\limits_{0}^{s}\int\limits_{\eta}^{\pi}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi\tau^{\rho+1}}d\xi d\tau\right). \tag{31}$$

$$J_8=\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}s^{[r]+1}\int\limits_{s}^{\pi}\int\limits_{0}^{t}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi^{[r]+1}\tau}d\xi d\tau=$$

$$=s^{[r]+1}\eta^{\rho}\int\limits_{s}^{\pi}\frac{d\xi}{\xi^{[r]+2}}\int\limits_{0}^{\eta}\frac{\omega_{f}^{[r]+1,[\rho]+1}(\xi,\tau)}{\tau}d\tau\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}+$$

$$+s^{[r]+1}\eta^{\rho}\int\limits_{s}^{\pi}\frac{d\xi}{\xi^{[r]+2}}\int\limits_{\eta}^{\pi}\frac{\omega_f^{[r]+1,[\rho]+1}(\xi,\tau)}{\tau}d\tau\int\limits_{\tau}^{\pi}\frac{dt}{t^{\rho+1}}\le$$

$$\le Cs^{[r]+1}\left(\int\limits_{s}^{\pi}\int\limits_{0}^{\eta}\frac{\omega_f^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi^{[r]+2}\tau}d\xi d\tau+\eta^{\rho}\int\limits_{s}^{\pi}\int\limits_{\eta}^{\pi}\frac{\omega_f^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi^{[r]+2}\tau^{\rho+1}}d\xi d\tau\right). \quad (32)$$

Analogously to estimate (32) we obtain

$$J_9=\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{dt}{t^{\rho+1}}t^{[\rho]+1}\int\limits_{0}^{s}\int\limits_{t}^{\pi}\frac{\omega_f^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi\tau^{[r]+2}}d\xi d\tau\le$$

$$\le C\eta^{\rho}\int\limits_{0}^{s}\int\limits_{\eta}^{\pi}\frac{\omega_f^{[r]+1,[\rho]+1}(\xi,\tau)}{\xi\tau^{[\rho]+1}}d\xi d\tau. \quad (33)$$

The other addends are established analogously. We estimate $\left\|\tilde{f}\right\|_C$. The following lemma is valid.

Lemma 1. *The following estimate is valid under the conditions of theorem 5.*

$$\left\|\tilde{f}\right\|_C\le C\left(\int\limits_{0}^{\pi}\int\limits_{0}^{\pi}\frac{\omega_f^{r,\rho}(s,t)}{st}dsdt+\int\limits_{0}^{\pi}\frac{\omega_f^{r,0}(s)}{s}ds+\int\limits_{0}^{\pi}\frac{\omega_f^{0,\rho}(t)}{t}dt+\|f\|_C\right). \quad (34)$$

Proof. We represent the integral $\tilde{f}(x,y)$ in the form

$$\tilde{f}(x,y)=\frac{1}{4\pi^2}\int\limits_{-\pi}^{\pi}g(y,x+s)\,ctg\frac{s}{2}ds\ , \quad (35)$$

where

$$g(y,x+s)=\int\limits_{-\pi}^{\pi}f(x+s,y+t)\,ctg\frac{t}{2}dt. \quad (36)$$

Then we have

$$\left|\tilde{f}(x,y)\right|=\frac{1}{4\pi^2}\left|\int\limits_{-\pi}^{\pi}\left[g(y,x+s)-g(y,x)\right]ctg\frac{s}{2}ds\right|\le 2\frac{1}{4\pi^2}\int\limits_{0}^{\pi}\frac{\omega_g^{0,1}(t)}{t}dt\ ,$$

Applying theorem 1 to $\omega_f^{0,1}(t)$ we obtain

$$\left|\tilde{f}(x,y)\right|\le C\left(\int\limits_{0}^{\pi}\frac{\omega_g^{0,[r]+1}(s)}{s}ds+\|g\|_C\right). \quad (37)$$

We estimate $\omega_g^{0,[r]+1}(s)$ and $\|g\|_C$.

$$g(y,x)=\int_{-\pi}^{\pi} f(x,y+t)\,ctg\frac{t}{2}dt=\int_{-\pi}^{\pi}\left[f(x,y+t)-f(x,y)\right]ctg\frac{t}{2}dt\Longrightarrow$$

$$\Longrightarrow \Delta_h^{0,[r]+1}g(y,x)=\int_{-\pi}^{\pi}\Delta_h^{[r]+1,0}\left[f(x,y+t)-f(x,y)\right]ctg\frac{t}{2}dt\Longrightarrow$$

$$\Longrightarrow \left|\Delta_h^{0,[r]+1}g(x,y)\right|\le 2\int_0^{\pi}\frac{\omega_f^{[r]+1,1}(h,t)}{t}dt\ .$$

Here passing to the supremum at $|h|\le s$ we have

$$\omega_g^{0,[r]+1}(s)\le C\int_0^{\pi}\frac{\omega_f^{[r]+1,1}(s,t)}{t}dt\le$$

$$\le C\left(\int_0^{\pi}\frac{\omega_f^{[r]+1,[\rho]+1}(s,t)}{t}dt\ +\omega_f^{[r]+1,0}(s)\right). \tag{38}$$

For $\|g\|$ analogously to the estimation $\left\|\tilde{f}\right\|_C$ we obtain

$$\|g\|_C\le C\left(\int_0^{\pi}\frac{\omega_f^{0,[\rho]+1}(t)}{t}dt+\|f\|_C\right). \tag{39}$$

From estimations (37)-(39) we have estimation (34). The lemma is proved.

Allowing for lemma 1 and putting the obtained estimation in (19) for $\omega_{\tilde{f}}^{[r]+1,\rho}(s,\eta)$ we complete the proof of estimation (18).

The theorem is proved.

Denote by $\Phi(0,\pi]$ a class of the functions φ determined on $(0,\pi]$ and having the properties: $\varphi\in C(0,\pi]$, $\varphi(t)>0$ $\varphi(t)\uparrow(t\uparrow)$ and $\varphi(t)\to 0$ $(t\to+0)$. Further, let

$$\Phi^r(0,\pi]=\{\varphi\in\Phi(0,\pi]|\ 0<t_1<t_2\le\pi\Longrightarrow t_1^r\varphi(t_2)\le C_\varphi t_2^r\varphi(t_1)\}$$

be a class of functions of continuous modulus type of the r-th order.

Denote by $\Phi^{r,\rho}(0,\pi]\times(0,\pi]$ a class of the functions $\omega(\delta,\eta)$ determined on $(0,\pi]^2$ and belonging to $\Phi^r(0,\pi]$ by the first and $\Phi^\rho(0,\pi]$ by the second argument. Let $\varphi\in\Phi^{r,\rho}(0,\pi]\times(0,\pi]$. Denote by

$$Z_\varphi^{r,\rho}=\left\{f\in C_{T^2}|\ \omega_f^{r,\rho}(\delta,\eta)=O(\varphi(\delta,\eta))\right\}$$

Introduce the norm $\|f\|_{Z_\varphi^{r,\rho}}=\max\left\{\|f\|_C,\ \sup\limits_{\delta,\eta\in(0,\pi]}\dfrac{\omega_f^{r,\rho}(\delta,\eta)}{\varphi(\delta,\eta)}\right\}$ in $Z_\varphi^{r,\rho}$.

By the traditional scheme we can prove that in this norm $Z_\varphi^{r,\rho}$ is a Banach space.

The following theorem holds

Theorem 6. *Let* $\varphi \in \Phi^{r,\rho}(0,\pi] \times (0,\pi]$. *If*

$$\int_0^\delta \int_0^\eta \frac{\varphi(s,t)}{st} dsdt + \delta^r \int_\delta^\pi \int_0^\eta \frac{\varphi(s,t)}{s^{r+1}t} dsdt + \eta^\rho \int_0^\delta \int_\eta^\pi \frac{\varphi(s,t)}{st^{\rho+1}} dsdt +$$

$$+\delta^r \eta^\rho \int_\delta^\pi \int_\eta^\pi \frac{\varphi(s,t)}{s^{r+1}t^{\rho+1}} dsdt = O(\varphi(\delta,\eta)) \ , \qquad (40)$$

then singular operator (26) acts from $Z_\varphi^{r,\rho}$ *to* $Z_\varphi^{r,\rho}$ *and is bounded.*

Proof. Let $f \in Z_\varphi^{r,\rho}$, $\varphi \in \Phi^{r,\rho}(0,\pi] \times (0,\pi]$ and condition (29) be satisfied. Then $\omega_f^{r,\rho}(\delta,\eta) \le \varphi(\delta,\eta) \|f\|_{Z_\varphi^{r,\rho}}$. From here and from conditions (40) it follows that all the conditions of theorem 5 are satisfied as a result of which the estimation

$$\omega_{\tilde f}^{r,\rho}(\delta,\eta) \le C \|f\|_{Z_\varphi^{r,\rho}} \left(\int_0^\delta \int_0^r \frac{\varphi(s,t)}{st} dsdt + \delta^r \int_\delta^\pi \int_0^\eta \frac{\varphi(s,t)}{s^{r+1}t} dsdt + \right.$$

$$+\eta^\rho \int_0^\delta \int_\eta^\pi \frac{\varphi(s,t)}{st^{\rho+1}} dsdt + \delta^r \eta^\rho \int_\delta^\pi \int_\eta^\pi \frac{\varphi(s,t)}{s^{r+1}t^{\rho+1}} dsdt + \eta^\rho \int_0^\delta \frac{\varphi(s,\pi)}{s} ds +$$

$$\left. +\delta^r \int_0^\eta \frac{\varphi(\pi,t)}{t} dt + \delta^r \eta^\rho \int_\delta^\pi \frac{\varphi(s,\pi)}{s^{r+1}} ds + \delta^r \eta^\rho \int_\eta^\pi \frac{\varphi(\pi,t)}{t^{\rho+1}} dt \right) \qquad (41)$$

holds.

We show that the expression in the parentheses in the right hand side of (41) is $O(\varphi(\delta,\eta))$. The sum of the first four addends by condition is $O(\varphi(\delta,\eta))$. We estimate $\eta^\rho \int_0^\delta \frac{\varphi(s,\pi)}{s} ds$. Consider two cases:

1. $0 < \eta \le \frac{\pi}{2}$. In this case we have

$$\eta^\rho \int_0^\delta \int_\eta^\pi \frac{\varphi(s,t)}{st^{\rho+1}} dsdt \ge \frac{\eta^\rho}{\pi} \int_0^\delta \int_\eta^\pi \frac{\varphi(s,t)}{st^\rho} dsdt \ge C\eta^\rho \int_0^\delta \frac{\varphi(s,\pi)}{s} ds \int_\eta^\pi dt \ge$$

$$\ge C\eta^\rho \int_0^\delta \frac{\varphi(s,\pi)}{s} ds \Longrightarrow \eta^\rho \int_0^\delta \frac{\varphi(s,\pi)}{s} ds = O(\varphi(\delta,\eta)) \qquad (42)$$

2. $\frac{\pi}{2} < \eta \le \pi$. We have

$$\int_0^\delta \int_0^\eta \frac{\varphi(s,t)}{st} dsdt \ge 2^{-\rho} \int_0^\delta \frac{\varphi(s,\eta)}{s\eta^\rho} ds \int_0^\eta t^{\rho-1} dt \ge \frac{1}{\rho 2^{3\rho}} \int_0^\delta \frac{\varphi(s,\pi)}{s} ds \Longrightarrow$$

$$\Longrightarrow \eta^{\rho}\int\limits_{0}^{\delta}\frac{\varphi\left(s,\pi\right)}{s}ds=O\left(\varphi\left(\delta,\eta\right)\right)\ . \qquad (43)$$

Analogously,

$$\delta^{r}\int\limits_{0}^{\eta}\frac{\varphi\left(\pi,t\right)}{t}dt=O\left(\varphi\left(\delta,\eta\right)\right)\ . \qquad (44)$$

We now consider the expression $\delta^{r}\eta^{\rho}\int\limits_{\delta}^{\pi}\frac{\varphi\left(s,\pi\right)}{s^{r+1}}ds$.

1. $0<\eta\leq\frac{\pi}{2}$. In this case we obtain

$$\delta^{r}\eta^{\rho}\int\limits_{\delta}^{\pi}\int\limits_{\eta}^{\pi}\frac{\varphi\left(s,t\right)}{s^{r+1}t^{\rho+1}}dsdt\geq\frac{2^{-\rho}\delta^{r}\eta^{\rho}}{\pi^{\rho+1}}\int\limits_{\delta}^{\pi}\frac{\varphi\left(s,\pi\right)}{s^{r+1}}ds\int\limits_{\eta}^{\pi}dt\Longrightarrow$$

$$\Longrightarrow \delta^{r}\eta^{\rho}\int\limits_{\delta}^{\pi}\frac{\varphi\left(s,\pi\right)}{s^{r+1}}ds\leq 2^{\rho+1}\pi^{\rho}\delta^{r}\eta^{\rho}\int\limits_{\delta}^{\pi}\int\limits_{\eta}^{\pi}\frac{\varphi\left(s,t\right)}{s^{r+1}t^{\rho+1}}dsdt=O\left(\varphi\left(\delta,\eta\right)\right)\ . \qquad (45)$$

2. $\frac{\pi}{2}<\eta\leq\pi$ and in this case we can prove that

$$\delta^{r}\eta^{\rho}\int\limits_{\delta}^{\pi}\frac{\varphi\left(s,\pi\right)}{s^{r+1}}ds\leq C\delta^{r}\int\limits_{\delta}^{\pi}\int\limits_{0}^{\eta}\frac{\varphi\left(s,t\right)}{s^{r+1}t}dsdt=O\left(\varphi\left(\delta,\eta\right)\right)\ . \qquad (46)$$

Analogously, we have

$$\delta^{r}\eta^{\rho}\int\limits_{\eta}^{\pi}\frac{\varphi\left(\pi,t\right)}{t^{\rho+1}}dt=O\left(\varphi\left(\delta,\eta\right)\right)\ . \qquad (47)$$

From estimations (42)-(47) it follows that

$$\omega_{\tilde{f}}^{r,\rho}\left(\delta,\eta\right)\leq C\left\|f\right\|_{Z_{\varphi}^{r,\rho}}\varphi\left(\delta,\eta\right) \qquad (48)$$

Thus, we proved that $\tilde{f}\in Z_{\varphi}^{r,\rho}$.

The boundedness of action of the operator S, i.e., the relation

$$\left\|\tilde{f}\right\|_{Z_{\varphi}^{r,\rho}}\leq C\left\|f\right\|_{Z_{\varphi}^{r,\rho}}$$

follows from estimate (48) and lemma 1. The theorem is proved.

References

[1]. Cesari L. *Sulle cerie di Fourier della fonzioni Lipshitziane di piu varable.* Ann. Scoul norm. super. Pisa Sci. fis. e mat., 1938, v.7, pp.279-295.

[2]. Jack I.E. *On one L.Cezari theorem on two-variable conjugate functions.* DAN SSSR, 1952, v.87, No6, pp.877-880. (Russian)

[3]. Bari N.K., Stechkin S.B. *The best approximations and differential properties of two conjugate functions.* Trudy mosk. math. obshestva. 1956, v.5, pp.483-522. (Russian)

[4]. Dzhvarsheyshvili A.G. *On one A.Zygmund inequality for two-variable functions.* Soobsheniye AN Gruz. SSR, 1954, v.15, No9. (Russian)

[5]. Musayev B.I., Salayev V.V. *On multivariable conjugate functions.* Ucheniye zapiski Azerb. Unniv., ser. phys.-math. sc., 1978, No4, pp.5-18. (Russian)

[6]. Bugrov Ya. S. *Fractional difference operators and classes of fucntions.* Trudy MIAN SSSR, 1985, v.172, pp.60-70. (Russian)

[7]. Zygmund A. *Trigonometric series.* M.: "Mir", 1975, v.1. (Russian)

[8]. Butzer P.L., Dyckhoff H., Gorlich E. and Stens R.L. *Best trigonometric approximation, fractional order derivatives and Lipshitz classes.* J. Canadien de Mathematiques, v.29, No4, pp.781-793.

[9]. Marchoud A. ,*Sur les derivatives et sur les differences des fonctions de variables reeles.* Tourn. Mathem. Pures et appl. (9), 6, 1927.

[10]. Abdullayev F.A. *On some properties of two-variable conjugate functions in smoothness modules.* Izvestiya VUZ-ov, Math., 1983, v.259, No12. (Russian)

[11]. Abdullayev F.A., Guliyev R.D. *Some estimations of singular integral in terms of fractional order smoothness modulus.* Dokl. NANA, v.LVI, No4-6, pp.57-64. (Russian)

[12]. Kakichev V.A. *Swartz integral and Hilbert formula for multi-complex variable analytical functions.* Izv. VUZ-ov, Math., 1959, No2. (Russian)

[13]. Gabdulkhaev B.G. *Approximated solutions of multivariate singular integral equations.* II Izv. VUZ-ov, Math., 1976, No1. (Russian)

Fuad A. Abdullayev
Baku States University.
23, Z.I.Khalilov str., AZ1148, Baku, Azerbaijan.
Tel.: (99412) 468 91 51 (apt.)

Received November 10, 2003; Revised April 29, 2004.
Translated by Mirzoyeva K.S.