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OPTIMAL CONTROL BY THE COEFFICIENTS OF A PARABOLIC EQUATION

Abstract

In this paper a problem on the optimal control by the coefficients of a parabolic type equation is considered. The correctness problems of statement of problem are investigated, the differentiability of objective functional is proved, the expression for its gradient is obtained, the necessary optimality condition of control is established.

The problems on optimal control problems by the coefficients of equations of mathematical physics are of great applied significance [1-3]. By studying the correctness of statement of these problems and by obtaining the necessary optimality conditions for them, difficulties concerned with their strong nonlinearity arise. These problems are nonlinear even when the equation describing the system status is linear, and the minimized functional is linear.

In this paper the optimal control problem by the coefficients of a parabolic equation is investigated. The similar problems earilear were studied in the papers [1,4-6] and others, provided that the coefficients of operation are found out in the spaces L_∞ and W_∞. In the present paper the coefficients of a parabolic equation are sought in the spaces L_q and W_p^1, where p and q are some finite numbers.

For the optimal control problem considered below, the problems of correctness and statement are investigated, the differentiability of objective functional is proved, the expression for its gradient is obtained and the necessary condition for optimal control is established.

1. Statement of problem. Let Ω be a bounded domain of two dimensional Euclidean space E_2 satisfying the condition of the cone [7, p.93], Γ be a boundary of the domain Ω which is issumed to be continuous by Lipschitz, $T>0$ be a given number, $0\le t\le T,\ Q_T=\Omega\times(0,T)\,,\ S_T=\Gamma\times[0,T],\ x=(x_1,x_2)$ be an arbitrary point of the domain Ω. The functional spaces $C(\bar{Q}_T),\ L_q\left(Q_T\right),\ \overset{\circ}{W}{}_2^1(\Omega),\ W_p^{1,1}(Q_T)$, $W_2^{1,1}(Q_T),\ W_2^{2,1}(Q_T), \overset{\circ}{V}{}_2^{1,0}(Q_T), (p,q\ge 1)$ used bellow are determined, for example, in [7].

Let the state of controlled process be described by the following parabolic type linear equation

$$\frac{\partial u}{\partial t}-\sum_{\alpha-1}^{2}\frac{\partial}{\partial x_\alpha}\left(\upsilon_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right)+\upsilon_0\left(x,t\right)u=f\left(x,t\right),$$

$$(x,t)\in Q_T, \tag{1}$$

where $f(x,t)$ is a given function from

$$L_2\left(Q_T\right),\ \upsilon=\upsilon(x,t)=\left(\upsilon_0(x,t),\ \upsilon_1(x,t),\ \upsilon_2(x,t)\right)$$

is a control, $u=u(x,t)$ is a solution of equation (1).

Let the following boundary conditions be given for equation (1)

$$u\,|_{t=0}=\varphi_0(x),\ x\in\Omega;\ u\,|_{S_T}=0, \tag{2}$$

where $\varphi_0(x)$ is a given function from $\overset{\circ}{W}{}_2^1(\Omega)$.

Let the control $\upsilon=\upsilon(x,t)$ be found in the following set of feasable controls

$$V=V_0\times V_1\times V_2\subset B\equiv L_q(Q_T)\times W_p^{1,1}(Q_T)\times W_p^{1,1}(Q_T), \tag{3}$$

where

$$V_0=\left\{\upsilon_0=\upsilon_0(x,t):\upsilon_0\in L_q(Q_T),\|\upsilon_0\|_{L_q(Q_T)}\leq d_0\right\}, \tag{4}$$

$$V_\alpha=\{\upsilon_\alpha=\upsilon_\alpha(x,t):\upsilon_\alpha\in W_p^{1,1}(Q_T),\ 0<\nu_\alpha\leq\upsilon_\alpha\leq\upsilon_\alpha(x,t)\leq$$
$$\leq\mu_\alpha \text{ a.e. } Q_T,\ \left\|\frac{\partial\upsilon_\alpha}{\partial t}\right\|_{L_p(Q_T)}\leq d_\alpha,\left\|\frac{\partial\upsilon_\alpha}{\partial x_i}\right\|_{L_p(Q_T)}\leq d_i^{(\alpha)},$$
$$i=1,2\},\alpha=1,2. \tag{5}$$

Here $d_0,\mu_\alpha,\nu_\alpha,d_i^{(\alpha)},d_\alpha(i,\alpha=1,2),\ p\geq 4,\ q>2$ are given numbers.

Consider the minimization problem of the functional.

$$J(\upsilon)=\|u(x,T;\upsilon)-\varphi_1(x)\|^2_{L_2(\Omega)} \tag{6}$$

on the solutions $u=u(x,t)=u(x,t;\upsilon)$ of boundary value problem (1), (2) of corresponding to all feasible controls $\upsilon\in V$. Below this problem is called problem (1)-(6).

Under the solution of boundary value problem (1), (2) for each $\upsilon\in V$ we shall understand the function $u=u(x,t)=u(x,t;\upsilon)$ from $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ satisfying the identity

$$\int\limits_{Q_T}\left(-u\frac{\partial\eta}{\partial t}+\sum_{\alpha-1}^{2}\upsilon_\alpha\frac{\partial u}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\alpha}+\upsilon_0u\eta\right)dxdt=$$
$$=\int\limits_\Omega\varphi(x)\eta(x.0)dx+\int\limits_{Q_T}f\eta dxdt, \tag{7}$$

for any function $\eta=\eta(x,t)$ from $\overset{\circ}{W}{}_2^1(\Omega)$ equal to zero at $t=T$. At accepted above assumptions from the results of the book [7, ch.III. 4] it follows that at each $\upsilon\in V$ boundary value problem (1), (2) has a unique solution from $V_2^{1,0}(\Omega_T)$ and the estimation

$$\|u\|_{V_2^{1,0}(Q_T)}\leq M\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{L_2(\Omega)}\right] \tag{8}$$

is valid.

Here and everywhere below M denotes positive constants which are independent of feasible controls and estimated quantities.

Besides, the solution of problem (1), (2) from $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ belongs also to space $W_{2,0}^{2,1}(Q_T)=W_2^{2,1}(Q_T)\cap\overset{\circ}{W}{}_2^{1,0}(Q_T)$, satisfying equation (1) at a.e. $(x,t)\in Q_T$ and the estimation [7. ch. III, 6]

$$\|u\|_{W_2^{2,1}(Q_T)}\leq M\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}\right] \tag{9}$$

holds.

2. Correctness of statement of the problem.

Theorem 1. *Let the conditions at statement of problem (1)-(6) be satisfied. Then in problem (1)-(6) the set of optimal controls* $V_* = \left\{ \upsilon_* \in V : J(\upsilon_*) = \inf_{\upsilon \in V} J(\upsilon) \right\}$ *is nonempty, weakly compact in* B *and any minimizing sequence* $\{\upsilon^{(n)}\}$ *weakly in* B *converges to* V_*.

Proof. We show that the functional $J(\upsilon)$ is weakly continuous on V. Let $\upsilon = (\upsilon_0, \upsilon_1, \upsilon_2) \in V$ be some control, $\left\{ \upsilon^{(n)} = \left(\upsilon_0^{(n)}, \upsilon_1^{(n)}, \upsilon_2^{(n)} \right) \right\} \subset V$ be an arbitrary sequence such that

$$\upsilon^{(n)} \to \upsilon \text{ weekly in } B. \tag{10}$$

Let $u^{(n)} = u^{(n)}(x,t) = u\left(x,t;\upsilon^{(n)}\right)$ be a solution of boundary value problem (1), (2) from $W_{2,0}^{2,1}(Q_T)$ at $\upsilon = \upsilon^{(n)}$. Then from estimation (9) it follows that

$$\left\| u^{(n)} \right\|_{W_2^{2,1}(Q_T)} \leq const, \forall n = 1,2,....) \tag{11}$$

Hence, from (10) and from compactness of embeddings $W_p^{1,1}(Q_T) \to C(\bar{Q}_T)$, $W_2^{2,1}(Q_T) \to L_r(Q_T), \forall r \in (0,\infty)$, $W_2^{2,1}(Q_T) \to L_r(\Omega)$ [7, p.78], [8, p.33] it follows that from the sequence $\left\{ \upsilon^{(n)}, u^{(n)} \right\}$ we can extract such subsequence which we denote by $\left\{ \upsilon^{(n)}, u^{(n)} \right\}$ that

$$\upsilon_0^{(n)} \to \upsilon_0 \text{ weekly in } L_q(Q_T), \tag{12}$$

$$\upsilon_\alpha^{(n)} \to \upsilon_\alpha (\alpha = 1,2) \text{ weekly in } W_p^{1,1}(Q_T) \text{ and strongly in } C(\bar{Q}_T), \tag{12b}$$

$$u^{(n)} \to u \text{ weekly in} W_2^{2,1}(Q_T) \text{ and strongly in } L_r(Q_T),\ \forall r \in (0,\infty), \tag{12c}$$

$$u^{(n)}\left|_{t=T} \to u\right|_{t=T} \text{ strongly in } L_2(\Omega), \tag{12d}$$

where $u = u(x,t)$ is some function from $W_{2,0}^{2,1}(Q)$. We show that $u = u(x,t) = u(x,t;\upsilon)$, i.e. the function $u = u(x,t)$ is a solution of problem (1), (2) at $\upsilon \in V$. The functions $u^{(n)} = u^{(n)}(x,t),\ n = 1,2,...$ satisfy the identity

$$\int_{Q_T} \left(-u^{(n)} \frac{\partial \eta}{\partial t} + \sum_{\alpha=1}^{2} \upsilon_\alpha^{(n)} \frac{\partial u^{(n)}}{\partial x_\alpha} \frac{\partial \eta}{\partial x_\alpha} + \upsilon_0^{(n)} n^{(n)} \eta \right) dxdt =$$

$$= \int_{\Omega} \varphi(x)\eta(x,0)dx + \int_{Q_T} f\eta dxdt, n = 1,2,..., \tag{13}$$

$$\forall \eta = \eta(x,t) \in \mathring{W}_2^{1,1}(Q_T),\ \eta(x,T) = 0.$$

Using (12 b), (12 c) and (11) we obtain that

$$\int_{Q_T} \upsilon_\alpha^{(n)} \frac{\partial u^{(n)}}{\partial x_\alpha} \frac{\partial \eta}{\partial x_\alpha} dxdt \to \int_{Q_T} \upsilon_\alpha \frac{\partial u}{\partial x_\alpha} \frac{\partial \eta}{\partial x_\alpha} dxdt, (\alpha = 1,2). \tag{14}$$

Besides, it is clear that

$$\int\limits_{Q_T} \upsilon_{\alpha}^{(n)} u^{(n)} \eta dxdt = \int\limits_{Q_T} \upsilon_0^{(n)} u\eta dxdt + \int\limits_{Q_T} \upsilon_0^{(n)} (u^{(n)} - u)\eta dxdt. \tag{15}$$

Since $u \in L_{2q/(q-2)}(Q_T)$, $\eta \in L_2(Q_T)$ then $u\eta \in L_{q(q-1)}(Q_T)$. Hence, from (12) it follows that

$$\int\limits_{Q_T} \upsilon_0^{(n)} u\eta dxdt \to \int\limits_{Q_T} \upsilon_0 u\eta dxdt. \tag{16}$$

Further, using the known inequality (1.8) from [7, p.75], the condition $\|\upsilon_0\|_{L_q(Q_T)} \leq d_0$ and (12b) we have

$$\left| \int\limits_{Q_T} \upsilon_0^{(n)} (u^{(n)} - u)\eta dxdt \right| \leq d_0 \left\| u^{(n)} - u \right\|_{L_{2q}/(q-2)(Q_T)} \times$$

$$\times \|\eta\|_{L_2(Q_T)} \to 0.$$

Allowing for these relations and (14) in (15) we obtain

$$\int\limits_{Q_T} \upsilon_0^{(n)} u^{(n)} \eta dxdt \to \int\limits_{Q_T} \upsilon_0 u\eta dxdt. \tag{17}$$

Now passing to the limit in (13) as $n \to \infty$ and allowing for (14), (17) and (12 b) we obtain that the function $u = u(x,t)$ satisfies identity (7). Hence and from $u \in W_{2,0}^{2,1}(Q_T)$ it follows that the function $u = u(x,t)$ satisfies equation (1) at a.e. $(x,t) \in Q_T$ and boundary conditions (2) are satisfied. Thus $u = u(x,t) = u(x,t;\upsilon)$.

Finally, using (22) from (1) we obtain that $J\left(\upsilon^{(n)}\right) \to J(\upsilon)$, i.e, the functional $J(\upsilon)$ is weakly continuous on V. Besides, the set V defined by relations (3), (5) is convex, closed and bounded in the reflexive Banach space B. Therefore, the statement of theorem 1 follows from the Weierstrass theorem [9, p.49]. The theorem is proved.

We now consider the minimization problem of the functional

$$I_\beta(\upsilon) = J(\upsilon) + \beta \|\upsilon - \omega\|_B^2 \tag{18}$$

on the set V defined by relations (3)-(5) under conditions (1), (2) where $\beta \geq 0$ is a given number, $\omega = (\omega_0, \omega_1, \omega_2) \in B$ is a given element, the functional $J(\upsilon)$ is determined by formula (6). We shall call this problem (1)-(5), (18).

Theorem 2. *Let the conditions of theorem 1 be satisfied and* $\beta \geq 0$. *Then for any* $\omega \in B$ *problem (1)-(5), (18) has at least one solution. If* $\beta > 0$ *then there exists dense subset* G *of the space* B *such that for any* $\omega \in G$, *problem (1)-(5), (18) has a unique solution.*

Proof. The functional $I_\beta(\upsilon)$ represents a sum of weakly continuous functional $J(\upsilon)$ and weakly semi-continuous below functional $\beta \|\upsilon - \omega\|_B^2$, $(\beta \geq 0)$. Consequently, the functional $I_\beta(\upsilon)$ is weakly semi-continuous below on V. Then from Weierstrass theorem [99 p. 49] it follows that at $\beta \geq 0$ problem (1)-(5), (18) has at least one solution.

Now let $\beta > 0$. As was proved in theorem 1 the functional $J(\upsilon)$ is weakly continuous on V. Therefore $J(\upsilon)$ is continuous by the norm of the B on V. Besides, the functional $J(\upsilon)$ on V is bounded from below, the space B is uniformly convex and the set V is closed bounded in B. Then by the known theorem [10], there exists a dense subset G of the space $\omega \in G$ such that for any $\beta > 0$ problem (1)-(5), (8) has a unique solution. Theorem 2 is proved.

3. Differentiability of functional and necessary optimality condition.

Let us introduce the following adjoint problem on defining the function $\psi = \psi(x,t) = \psi(x,t;\upsilon)$ from the conditions

$$\frac{\partial\psi}{\partial t} + \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\left(\upsilon_\alpha \frac{\partial\psi}{\partial x_\alpha}\right) - \upsilon_0\psi = 0,\ (x,t) \in Q_T, \tag{19}$$

$$\psi\,|_{t=T} = -2[u,(xT;\upsilon) - \varphi_1(x)], x \in \Omega,\ \psi\,|_{S_T} = 0, \tag{20}$$

where $u = u(x,t;\upsilon)$ is a solution of problem (1), (2) at $\upsilon \in V$.

Under the solution of problem (19), (20) at each $\upsilon \in V$ we will understand the function $\psi = \psi(x,t)$ from $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ satisfying the identity

$$\int_{Q_T} \left(\psi \frac{\partial\eta}{\partial t} + \sum_{\alpha=1}^{2} \upsilon_\alpha \frac{\partial\psi}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\alpha} + \upsilon_0 u\eta\right) dxdt =$$

$$= -2\int_{Q_T} [u,(xT;\upsilon) - \varphi_1(x)]\eta(x,T)dx, \tag{21}$$

$$\forall\eta = \eta(x,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T),\ \eta(x,0) = 0.$$

If instead of variable t we take a new independent variable $\tau = T - t$ in relations (19), (20) then we obtain boundary value problem of type (1), (2). Therefore from the results of the book [7 ch.IV, 4] It follows that problem (19), (20) at each $\upsilon \in V$ has a unique solution from $\overset{\circ}{V}{}_2^{1,0}(Q_T)$. Moreover, if $\varphi_1(x) \in \overset{\circ}{W}{}_2^{1}(\Omega)$, then the solution of problem (19), (20) belongs to the space $W_{2,0}^{2,1}(Q_T)$, satisfies equation (19) at a.e. $(x,t) \in Q_T$, and the estimate [7.ch. III, §6]

$$\|\psi\|_{W_2^{2,1}(Q_T)} \le M\,\|u(x,T;\upsilon) - \varphi_1(x)\|_{W_2^1(\Omega)} \tag{22}$$

holds. Besides allowing for the inequality [11 p.161]

$$\|u(x,T;\upsilon)\|_{W_2^1(\Omega)} \le M\,\|u\|_{W_2^{2,1}(Q_T)}$$

and estimation (6), from (22) we obtain

$$\|\psi\|_{W_2^{2,1}(Q_T)} \le M\left[\|f\|_{L_2(Q_T)} + \|\varphi_0\|_{W_2^1(\Omega)} + \|\varphi_1\|_{W_2^1(\Omega)}\right]. \tag{23}$$

Now we introduce the following boundary value problem on defining the functions $\theta_i = \theta_i(x,t) = \theta_i(x,t;\upsilon)$ $(i = 1,2)$ from the conditions

$$-\sum_{\alpha=1}^{2} \frac{\partial^2\theta_i}{\partial x_\alpha^2} - \frac{\partial^2\theta_i}{\partial t^2} + \theta_i = \frac{\partial u}{\partial x_i}\frac{\partial\psi}{\partial x_i},\ (x,t) \in Q_T, \tag{24}$$

$$\frac{\partial\theta_i}{\partial\nu}\,|_{S_T} \equiv \sum_{\alpha=1}^{2}\frac{\partial\theta_i}{\partial x_\alpha}\cos(\nu,x_\alpha)\Big|_{S_T} = 0,$$

$$\frac{\partial\theta_i}{\partial t}\,|_{t=0} = \frac{\partial\theta_i}{\partial t}\,|_{t=T} = 0,\ x\in\Omega,\ (i=1,2), \tag{25}$$

where $u = u(x,t;\upsilon)$, $\psi = \psi(x,t;\upsilon)$ are solutions of problems (1), (2) and (19), (20) at $\upsilon \in V$, respectively ν is a unit exterior normal to Γ.

Under the solution of problem (24), (25) we we'll understand the function $\theta_i = \theta_i(x,t) = \theta_i(x,t;\upsilon)$ from $W_2^{1,1}(Q_T)$ satisfying the identity.

$$\int_{Q_T}\left(\sum_{\alpha=1}^{2}\frac{\partial\theta_i}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\alpha} + \frac{\partial\theta_i}{\partial t}\frac{\partial\eta}{\partial t} + \theta_i\eta\right)dxdt =$$

$$= \int_{Q_T}\frac{\partial u}{\partial x_i}\frac{\partial\psi}{\partial x_i}\eta dxdt,\ (i=1,2)\,. \tag{26}$$

Boundary value problem (24), (25) is a Neumann problem for elliptic equation (24).

According to lemma 3.3 from [7,p.95] the estimations

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L_4(Q_T)} \le M\,\|u\|_{W_2^{2,1}(Q_T)}\,,$$

$$\left\|\frac{\partial\psi}{\partial x_i}\right\|_{L_4(Q_T)} \le M\,\|\psi\|_{W_2^{2,1}(Q_T)}\,,\ (i=1,2) \tag{27}$$

are valid.

Hence, if follows that $\frac{\partial u}{\partial x_i}\frac{\partial\psi}{\partial x_i} \in L_2(Q_T), (i=1,2)$. Then from the Lax Milgram lemma [12, p.39]. It follows that problem (24), (25) at each $\upsilon \in V$ has a unique solution from $W_2^{1,1}(Q_T)$ and

$$\|\theta_i\|_{W_2^{1,1}(Q_T)} \le M\left\|\frac{\partial u}{\partial x_i}\right\|_{L_4(Q_T)}\left\|\frac{\partial\psi}{\partial x_i}\right\|_{L_4(Q_T)},\ (i=1,2)\,.$$

Then here allowing for estimations (27), (9) and (23) we have

$$\|\theta_i\|_{W_2^{1,1}(Q_T)} \le M\left[\|f\|_{L_2(Q_T)} + \|\varphi_0\|_{W_2^1(\Omega)}\right]\times$$

$$\times\left[\|f\|_{L_2(Q_T)} + \|\varphi_0\|_{W_2^1(\Omega)} + \|\varphi_1\|_{L_2(Q_T)}\right],\ (i=1,2). \tag{28}$$

Theorem 3. *Let the conditions of theorem 1 be satisfed and* $\varphi_1(x) \in \overset{\circ}{W}{}_2^1(\Omega)$. *Then functional (6) is continuously differentiable by Frechet on* V *and gradient has the form*

$$J'(\upsilon) = (u(x,t;\upsilon)\,\psi\,(x,t;\upsilon)\,,\theta_1\,(x,t;\upsilon)\,,$$

$$\theta\,(x,t;\upsilon)\,. \tag{29}$$

Proof. Let $\delta v=(\delta v_0,\delta v_1,\delta v_2)\in B$ be an increment of control on the element $v\in V$ such that $v+\delta v\in V$. Then the solution of problem (1), (2) gets the increment

$$\delta u(x,t)=u\left(x,t;v+\delta v\right)-u\left(x,t;v\right).$$

From conditions (1), (2) it follows that the function on $\delta u=\delta u\left(x,t\right)\in W_{2,0}^{2,1}(Q_T)$ is a solution of the problem

$$\frac{\partial\delta u}{\partial t}-\sum_{\alpha=1}^{2}\frac{\partial}{\partial x_\alpha}\left((v_\alpha+\delta v_\alpha)\frac{\partial\delta u}{\partial x_\alpha}\right)+\left(v_0+\delta v_0\right)\delta u=$$

$$=\sum_{\alpha=1}^{2}\frac{\partial}{\partial x_\alpha}\left(\delta v_\alpha\frac{\partial u}{\partial x_\alpha}\right)-\delta v_0 u,(x,t)\in Q_T, \tag{30}$$

$$\delta u\left|_{t=0}\right.=0,\ x\in\Omega;\ \delta u\left|_{S_T=0}\right. \tag{31}$$

and for it the estimation [7, ch. II, 6]

$$\|\delta u\|_{W_2^{2,1}(Q_T)}\le M\left[\sum_{\alpha=1}^{2}\left\|\delta v_\alpha\frac{\partial^2 u}{\partial x_\alpha^2}\right\|_{L_2(Q_T)}\right.$$

$$\left.+\left\|\frac{\partial\delta v_\alpha}{\partial x_\alpha}\frac{\partial u}{\partial x_\alpha}\right\|_{L_2(Q_T)}\right)+\|\delta v_0 u\|_{L_2(Q_T)} \tag{32}$$

is valid.

Using the boundedness of the embedding $W_p^{1,1}(Q_T)\to C\left(\bar{Q}_T\right)$ [7, p. 78], the Hōlder, inequality (27) and condition $p\ge 4$ we obtain

$$\left\|\delta v_\alpha\frac{\partial^2 u}{\partial x_\alpha^2}\right\|_{L_2(Q_T)}+\left\|\frac{\partial\delta v_\alpha}{\partial x_\alpha}\frac{\partial u}{\partial x_\alpha}\right\|_{L_2(Q_T)}\le$$

$$\le\|\delta v_\alpha\|_{C(\bar{Q}_T)}\left\|\frac{\partial^2 u}{\partial x_\alpha^2}\right\|_{L_2(Q_T)}+\left\|\frac{\partial^2\delta v_\alpha}{\partial x_\alpha}\right\|_{L_2(Q_T)}\left\|\frac{\partial u}{\partial x_\alpha}\right\|_{L_2(Q_T)}\le$$

$$\le M\|\delta v_\alpha\|_{W_p^{1,1}(Q_T)}\|u\|_{W_2^{2,1}(Q_T)},(\alpha=1,2). \tag{33}$$

Besides, using the Hōlder inequality, boundedness of embedding $W_2^{2,1}(Q_T)\to L_r(Q_T),\ \forall r\in(0,\infty)$, [8, p.33] and the condition $q>2$, we have

$$\|\delta v_0 u\|_{L_2(Q_T)}\le\|\delta v_0\|_{L_q(Q_T)}\|u\|_{L_{2q/(q-2)}}\le$$

$$\le M\|\delta v_0 u\|_{L_2(Q_T)}\|u\|_{W_2^{2,1}(Q_T)}. \tag{34}$$

Allowing for (35) and (34) in (32) and using (9) we obtain the estimation

$$\|\delta u\|_{W_2^{2,1}(Q_T)}\le M\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}\right]\|\delta v\|_B. \tag{35}$$

We now consider the increment of functional (1)

$$\delta J(v)=J\left(v+\delta v\right)-J(v)=2\int\limits_{Q_T}\left[u(x,T;v)-\varphi_1(x)\right]\times$$

$$\times\delta u(x,T)dx+\|\delta u(x,T)\|^2_{L_2(\Omega)}\,. \tag{36}$$

Using conditions (19), (20) and (30), (31) this expression is easily represented in the following form

$$\delta J(v)=\int\limits_{Q_T}\left(\sum_{i=1}^{2}\frac{\partial u}{\partial x_i}\frac{\partial\psi}{\partial x_i}\delta v_i+u\psi\delta v_0\right)dxdt+R(\delta v),$$

where

$$R(\delta v)=\int\limits_{Q_T}\left(\sum_{i=1}^{2}\frac{\partial\delta u}{\partial x_i}\frac{\partial\psi}{\partial x_i}\delta v_i+\delta u\psi\delta v_0\right)dxdt+$$

$$+\|\delta u(x,T)\|^2_{L_2(\Omega)}\,. \tag{37}$$

If we put $\eta=\delta v_i$ in (26) and allow for obtained equality in (36), then we have

$$\delta J(v)=\int\limits_{Q_T}\left[\sum_{i=1}^{2}\left(\theta_i\delta v_i+\frac{\partial\theta_i}{\partial t}\frac{\partial\delta v_i}{\partial t}+\sum_{\alpha=1}^{2}\frac{\partial\theta_i}{\partial x_\alpha}\frac{\partial\delta v_i}{\partial x_\alpha}\right)+u\psi\delta v_0\right]\times$$

$$\times dxdt+R(\delta v). \tag{38}$$

We now lead the estimation of the remainder term $R(\delta v)$. Using the Hōlder inequality, boundedness of embeddings

$$W_p^{1,1}(Q_T)\to C(\bar{Q}_T),W_2^{2,1}(Q_T)\to L_{2q/(q-2)}(Q_T)$$

$(q>2)$ [7, p. 78], [8, p. 33] and estimations (23), (35), we have

$$\left|\int\limits_{Q_T}\left(\sum_{i=1}^{2}\frac{\partial\delta u}{\partial x_i}\frac{\partial\psi}{\partial x_i}\delta v_i+\delta u\psi\delta v_0\right)dxdt\right|\le$$

$$\le\sum_{i=1}^{2}\|\delta v_i\|_{C(\bar{Q}_T)}\left\|\frac{\delta u}{\partial x_i}\right\|_{L_2(Q_T)}\left\|\frac{\delta\psi}{\partial x_i}\right\|_{L_2(Q_T)}+$$

$$+\|\delta v_0\|_{L_q(Q_T)}\|\delta v\|_{L_{2q/(q-2)}(Q_T)}\|\psi\|_{L_2(Q_T)}\le M\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}\right]\times$$

$$\times\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}+\|\varphi_1\|_{W_2^1(\Omega)}\right]\|\delta v\|^2_B\,.$$

Hence and from (35) it follows that for the remainder term $R(\delta v)$ determined by equality (32), the estimation

$$|R(\delta v)|\le M\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}\right]\times$$

$$\times\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}+\|\varphi_1\|_{W_2^1(\Omega)}\right]\|\delta v\|_B^2 \tag{39}$$

is valid.

Then from equality (38) and estimation (39) it follows that functional (6) is differentiable by Frenchet on V and for its gradient equality (29) is valid.

It remains to show that $v\to J'(v)$ is a continuous mapping from V in B^*, where $B^*\equiv L_{q/(q-1)}(Q_T)\times W^{1,1}_{p/(p-1)}(Q_T)\times W^{1,1}_{p/(p-1)}(Q_T)$ is a space adjoint to B.

Let $\delta\psi=\psi\left(x,t;v+\delta v\right)-\psi\left(x,t;v\right),\ \delta\theta_i=\theta_i(x,t;v+\delta v)-\theta_i(x,t;v)$ $(i=1,2)$ be increments of solutions of problems (19), (20) and (24), (25), respectively. Reasoning analogously as was obtained estimation (35) for the function δu and estimation (28) the function θ_i it is easily shown that for the functions $\delta\psi$ and $\delta\theta_i$ $(i=1,2)$ the estimations

$$\|\delta\psi\|_{W_2^{2,1}(Q_T)}\le M\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}+\|\varphi_1\|_{W_2^1(\Omega)}\right]\|\delta v\|_B\,, \tag{40}$$

$$\|\delta\theta_i\|_{W_2^{1,1}(Q_T)}\le M\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}\right]\times$$

$$\times\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}+\|\varphi_1\|_{W_2^1(\Omega)}\right]\left(\|\delta v\|_B+\|\delta v\|_B^2\right)\ (i=1,2) \tag{41}$$

are valid.

Then using equality (28) estimations (8), (23), (35), (40), (41) we obtain the estimations

$$\left\|J'\left(v+\delta v\right)-J'(v)\right\|_{B^*}\le M\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}\right]\times$$

$$\times\left[\|f\|_{L_2(Q_T)}+\|\varphi_0\|_{W_2^1(\Omega)}+\|\varphi_1\|_{W_2^1(\Omega)}\right]\left(\|\delta v\|_B+\|\delta v\|_B^2\right),$$

from which the continuity of $J'(v)$ on V follows.

Theorem 3 is proved.

Now we formulate a necessary optimality condition for solution of problem (1)-(6).

Theorem 4. *Let the conditions of theorem 3 be satisfied and*

$$v^*=\left(v_0^*(x,t),v_1^*(x,t),v_2^*(x,t)\right)\in V$$

be an optimal control for problem (1)-(6). Then the following inequality is satisfied

$$\int\limits_{Q_T}\left[\sum_{\alpha=1}^{2}\left[\theta_\alpha^*(x,t)v_2\left(x,t\right)-v_\alpha^*(x,t)\right)+\frac{\partial\theta_\alpha^*}{\partial t}\left(\frac{\partial v_\alpha}{\partial t}-\frac{\partial v_\alpha^*}{\partial t}\right)+\right.$$

$$\left.+\sum_{k=1}^{2}\frac{\partial\theta_\alpha^*}{\partial x_k}\left(\frac{\partial v_\alpha}{\partial x_k}-\frac{\partial v_\alpha^*}{\partial x_k}\right)\right]+u^*\left(x,t\right)\psi^*\left(x,t\right)\left(v_0(x,t)-v_0^*(x,t)\right)\Bigg\}\,dxdt\ge 0 \tag{42}$$

for any function $v=(v_0(x,t),v_1(x,t),v_2(x,t))\in V$, where $u^*(x,t);\psi^*(x,t)$ and $\theta_\alpha^*(x,t)$ $(\alpha=1,2)$ are solutions of problems (1), (2); (19), (20); and (24), (25) at $v=v^*$ respectively.

Proof. According to theorem 3 the functional $J(\upsilon)$ is continuously differentiable by Frechet on V and for its gradient formula (29) is valid. The set V defined by relations (3)-(5) is convex. Then by the known theorem [9, p.28] on the element $\upsilon \in V^*$ delivering the minimum to the functional $J(\upsilon)$, it is necessary the fulfilment of the inequality

$$< J'(\upsilon^*), \upsilon - \upsilon^* >_B \geq 0,$$

for any $\upsilon \in V$. Hence and from (29) the validity of inequality (42) follows.

The theorem is proved.

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